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Global asymptotic stability of an umploment model using geometric approach

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Abstract

In this work, we study the global asymptotic stability of a nonlinear unemployment model. The nonlinearity comes from the matching process between vacancies and unemployed people. Thus, we assume that the employment rate is a general nonlinear function, which includes the bilinear form presented in the previous scientific research. We provide conditions that guarantee the existence and uniqueness of a positive equilibrium. To study the dynamic behavior of this equilibrium, we propose Li's geometrical approach. This technique ensures global asymptotic stability without the need to impose an additional condition. Finally, we provide numerical illustrations of our study.

Keywords: Unemployment model; Nonlinear employment rate; Global stability; Geometric approach

1. Introduction

Unemployment is a very complex phenomenon. It is defined as the surplus of people who have the capacity to work and who are actively seeking employment. This state occurs when the supply of labor exceeds the demand. Unemployment is becoming, by its magnitude, one of the most important social and economic problems in the world.

Unemployment has directly contributed to the development of sociological surveys and mathematical models intended to understand its evolution (for instance [1–6] and the references therein). In particular, the differential equations can explain the sharp increases in unemployment rates, by

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testing hypotheses and providing scenarios using numerical simulations. Then propose economic policy measures to reduce these rates. However, translating the phenomenon of unemployment into mathematical equations is not always easy. One of the important questions that can be encountered is how to mathematically describe the matching process between vacancies and unemployed? is it suitable by a linear function or by a nonlinear function? in the nonlinear case, what will be the form of this function?

Historically, the field of static analysis was the field of the first experiments to model the evolution of labor markets. In 1984, Albrecht and Axell [1] are developed a simple general equilibrium model of steady-state unemployment. The main result of their work shows that a general increase in unemployment benefits increases the equilibrium unemployment rate and that a selective increase decreases this equilibrium. Pissarides was interested in the theory of equilibrium unemployment [4–6]. Firstly, he used data on unemployment flows and job vacancies to study the phenomenal rise of unemployment in Britain. The author constructed the following model [6]:

$$u_{t+l} = s_t[1 + (1 - s_t - p_t) + (1 - s_t - p_t)^2] + (1 - s_t - p_t)^3 u_t.$$

where u_t is the unemployment rate at the beginning of quarter t, s_t is the ratio of the average monthly inflow for the three months ended at the beginning of $t + 1$ to the stock of employees in employment at the beginning of quarter t, p is the ratio of the average monthly outflows for the three months ended at the beginning of the quarter, $t + 1$ to the stock of unemployment at the beginning of quarter t. He concluded that changes in unemployment in Britain have been driven mainly by the changes in the rate at which unemployed workers move into employment. Secondly, he proposed one of the first model of unemployment evolution given by the following ordinary differential equation [4].

$$\frac{du}{dt} = \lambda(1 - u) - \theta q(\theta)u, \tag{1.1}$$

where u denote unemployment rate (fraction of unmatched workers), λ is the rate of job destruction, θ is the ratio of vacancies to unemployment and q is the recruitment rate. In this model, the matching process between vacancies and unemployed people is given by the nonlinear function $\theta q(\theta)u$, and the author proved that unemployment persists in the positive equilibrium.

In [3], Mortensen developed an unemployment equilibrium model as follows:

$$\begin{cases} \dfrac{dn}{dt} = h(p)(1 - n) - \delta n, \\ \dfrac{dp}{dt} = (r + \delta)p + g(p) - f(n), \end{cases} \tag{1.2}$$

where n is the current employment, p is the match surplus, h is the matching rate per unemployed workers, g is the net expected income derived, r is the discount rate, δ is rate of job separations, and f is the production function. The author studied the global dynamics of system (1.2), in the case of increasing returns to scale in production and constant returns to scale in the matching process. He proved the existence of multiple equilibria and guaranteed the existence of limit cycles. In [2], Daud and Ghozali are proposed a simple mathematical model of unemployment by considering two variables: the number of employed persons, denoted E, and the number of unemployed persons, denoted U. They showed that the model has only one non-negative equilibrium. The stability analysis performed using the Routh-Hurwitz criterion implies that the model has a locally asymptotically stable equilibrium point. Munoli et al. [7], included in Daud's model the variable V which denotes the number of vacancies, Munoli's model is described as follows:

$$\begin{cases} \dfrac{dU}{dt} = \Lambda - kUV + \gamma E - \alpha U, \\ \dfrac{dE}{dt} = kUV - (\alpha_2 + \gamma)E, \\ \dfrac{dV}{dt} = (\alpha_2 + \gamma)E - \delta V + \Phi U, \end{cases} \tag{1.3}$$

where Λ is the unemployment growth rate, k denotes the recruitment rate,α_1 is the sum of migration and death rates of unemployed people, α_2 is the sum of retirement and death rates of employed, γ is the rate of persons who are fired from their jobs, Φ is the rate of creating new vacancies and δ is the diminution rate of vacancies due to lack of funds.

In [8] Maalwi et al. analyzed the following version:

$$\begin{cases} \dfrac{dU}{dt} = \Lambda - kUV + \beta E - \mu U, \\ \dfrac{dE}{dt} = kUV - \beta E - \alpha E, \\ \dfrac{dV}{dt} = \alpha E - \delta V, \end{cases} \tag{1.4}$$

where β is the rate of resigned or fired for jobs, α the rate of jobs loss by migration, retirement or death, μ is the sum of migration and death rates of unemployed and δ is the rate of abolition of vacancies. The authors proved that, if the basic reproduction number is bounded, the employment-free equilibrium is unstable and the positive equilibrium is globally asymptotically stable. More recently, Petaratip et al. [9] incorporated a time delay in the process of creating vacancies. They found the same results, in [8], on the global asymptotic stability and they demonstrated that this model does not present a Hopf bifurcation. The most important feature in the above mentioned models was the use of a bilinear form ($k.UV$) for describing the matching process between vacancies and unemployed.

In this work, we propose to study the following model with a general function of matching process:

$$\begin{cases} \dfrac{dU}{dt} = \Lambda - m(U)V + \beta E - \mu U, \\ \dfrac{dE}{dt} = m(U)V - \beta E - \alpha E, \\ \dfrac{dV}{dt} = \sigma E - \delta V, \end{cases} \tag{1.5}$$

here m is the rate of employment, which describing the number of unemployed people becoming employed.

Let, m is continuously differentiable in the interior of $\mathbb{R}^2_+$ satisfying the following hypotheses.

(H_0): $m(0) = 0$,

(H_1): m is a monotone increasing function.

This generalized employment rate includes the constant employment and linear cases. Assumptions (H_0) and (H_1) mean that matching between vacancies increases as the number of unemployed increases, it cancels if there are no unemployed people.

This paper is organized as follows: section 2 gives conditions of the existence and uniqueness of the equilibria, section 3 describes the behavior of the trivial equilibrium, and section 4 propose local stability of the positive equilibrium, section 5 presents the global asymptotic stability of the positive equilibrium using Li's geometrical approach, and section 6 provides numerical illustrations of our results. Finally, section 7 summarizes our contribution and gives a comparison of our result with the existing one.

2. Equilibria

We consider the set,

$$T = \left\{(U,E,V), \quad 0 \le U + E \le \frac{\Lambda}{\kappa_1}, 0 \le V \le \kappa_2\right\}.$$

T is bounded and positively invariant, where $\kappa_1 = \min\{\alpha, \mu\}$, and $\kappa_2 = \frac{\sigma}{\delta\kappa_1}$. Using next matrix generation [20], we define the basic reproduction number:

$$R_0 = \frac{\sigma m\left(\frac{A}{\mu}\right)}{\delta(\alpha+\beta)}.$$

We prove the following result of the existence and uniqueness of trivial and non-trivial equilibriums.

Theorem 2.1. [13]

System (1.5) admits two equilibriums: a trivial equilibrium $P_0 = \left(\frac{A}{\mu}, 0, 0\right)$, and a positive equilibrium $P = (U^*, E^*, V^*)$ when $R_0 > 1$.

Proof. $(U,E,V))$ is an equilibrium for system (1.5) if and only if

$$\begin{cases} \Lambda - \mu U - m(U)V + \beta E = 0, \\ m(U)V - (\alpha+\beta)E = 0, \\ \sigma E - \delta V = 0. \end{cases} \tag{2.1}$$

If $E = 0$, then $V = 0$, and $U = \frac{A}{\mu}$, therefore we get $P_0 = \left(\frac{A}{\mu}, 0, 0\right)$. Otherwise, if $E \neq 0$, the system (1.5) becomes

$$\begin{cases} U = \frac{A}{\mu} - \frac{\alpha E}{\mu}, \\ V = \frac{\sigma E}{\delta}, \\ \frac{\sigma}{\delta} m\left(\frac{A}{\mu} - \frac{\alpha E}{\mu}\right) - (\alpha+\beta) = 0. \end{cases} \tag{2.2}$$

Let's pose

$$f(E) := \frac{\sigma}{\delta} m\left(\frac{\Lambda}{\mu} - \frac{\alpha E}{\mu}\right) - (\alpha+\beta).$$

f is continuous and strictly decreasing in interval $\left[0, \frac{\Lambda}{\mu}\right]$, $\left(m'(E) = \frac{-\alpha\sigma}{\mu\delta} f'\left(\frac{A}{\mu} - \frac{\alpha E}{\mu}\right) < 0\right)$, and $f(0) = \frac{\sigma}{\delta} m\left(\frac{\Lambda}{\mu}\right) - (\alpha+\beta) > 0$, if $R_0 > 1$.

According to the Intermediate Value Theorem, there is one and only one positive solution E^* of equation $f(E)=0$. This concludes the proof.

3. Global Stability of Equilibrium P_0

Using an appropriate Lyapunov function and the linearization of (1.5) around P_0, we prove the following results:

Theorem 3.1. [8, 11]

- If $R_0 < 1$, then P_0 is globally asymptotically stable;
- If $R_0 > 1$, then P_0 is unstable.

Proof. For a positive constant p, let $L_1(t)$ be the Lyapunov functional candidate namely.

$$L_1(t) = U - U_0 - U_0 \ln\left(\frac{U}{U_0}\right) + \frac{\beta}{2(\mu+\alpha)}(U - U_0 + E)^2 + E + pV. \tag{3.1}$$

Its time derivative is

$$\frac{dL_1(t)}{dt} = \left(1 - \frac{U_0}{U}\right)\dot{U} + \frac{\beta}{\mu+\alpha}(U - U_0 + E)(\dot{U} + \dot{E}) + \dot{E} + p\dot{V}.$$

Using system (1.5), we find

$$\begin{aligned}\frac{dL_1(t)}{dt} &= \left(1 - \frac{U_0}{U}\right)(\Lambda - m(U)V + \beta E - \mu U) + \frac{\beta}{\mu+\alpha}(U - U_0 + E)(\Lambda - \mu U - \alpha E)\\ &+ m(U)V - (\alpha+\beta)E + p(\sigma E - \delta V)\\ &= -\left[\frac{\mu}{U} + \frac{\beta E}{UU_0} + \frac{\mu\beta}{(\mu+\alpha)U_0}\right](U_0 - U)^2 - \frac{\alpha\beta}{(\mu+\alpha)U_0}E^2 + \sigma\left(p - \frac{\alpha+\beta}{\sigma}\right)E + \delta\left(\frac{m(U_0)}{\delta} - p\right)V.\end{aligned}$$

On the other hand, since $R_0 < 1$, we know that there exists $p > 0$ such that $\frac{m(U_0)}{\delta} < p < \frac{\alpha+\beta}{\sigma}$. Therefore P_0 is globally asymptotically stable. Let us now examine the instability of P_0. Let $J(P_0)$ be the Jacobian matrix of the, system (1.5) at P_0. Then

$$J(P_0) = \begin{pmatrix} -\mu & \beta & -m(U_0) \\ 0 & -\alpha-\beta & m(U_0) \\ 0 & \sigma & -\delta \end{pmatrix}$$

The eigenvalues of the matrix are $a_1 = -\mu < 0$, a_2 and a_3 such that

$$\lambda_2 + \lambda_3 = -(\alpha+\beta)\delta(1 - R_0).$$

If $R_0 > 1$, then $J(P_0)$ admits a positive eigenvalue and therefore P_0 is unstable.

4. Local stability of equilibrium P

In the following result, we examine the local stability of the positive equilibrium P of (1.5). This result play a very important part in the proof its global stability.

Theorem 4.1 *[13] Under Assumptions* (H_0) *and* (H_1).
If $R_0 > 1$, then the positive equilibrium P is locally asymptotically stable.

Proof. Let $J(P)$ denote the Jacobian matrix of system (1.5) at P. Then

$$J(P)=\begin{pmatrix} -\mu-m'(U^*)V^* & \beta & -m(U^*) \\ m'(U^*)V^* & -\alpha-\beta & m(U^*) \\ 0 & \sigma & -\delta. \end{pmatrix}$$

Its characteristic polynomial is:

$$\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0, \tag{4.1}$$

where

$$\begin{aligned} a_1 &= \alpha+\beta+\delta+\mu+m'(U^*)V^*, \\ a_2 &= \mu\delta+\delta(\alpha+\beta)+(\alpha+\delta)m'(U^*)V^*+\delta(\alpha+\beta)-\sigma m(U^*), \\ a_3 &= \alpha\delta m'(U^*)V^*-\mu(\delta(\alpha+\beta)-\sigma m(U^*)). \end{aligned}$$

By using the relations: $\alpha+\beta=\dfrac{m(U^*)V^*}{E^*}$ and $E^*=\dfrac{\delta V^*}{\sigma}$, we can easily obtain that

$$\delta(\alpha+\beta)-\sigma m(U^*)=0.$$

Since m is increasing, then

$$a_i>0, \qquad i=1,2,3,$$

and

$$a_1a_2-a_3>0.$$

Hence, by the Hurwitz's criterion, we have the local stability of P^* for $R_0>1$. This concludes the proof of Theorem 5.2.

5. Global Stability of the Positive Equilibrium

To our knowledge, the use of Lyapunov functions for the study of the global stability of the positive equilibrium P, of system (1.5), leads to impose an additional condition on the basic reproduction number. In this section, we use Li's geometric approach to avoid this additional condition [10, 12]. To prove this result, We start with the following proposition

Proposition 5.1. *System (1.5) is uniformly persistent if and only if* $R_0>1$.

Proof. According to Theorem, if $R_0>1$, then the trivial equilibrium $P_0=(\Lambda,0,0)$ is unstable. Hence, system (1.5) is uniformly persistent for $R_0>1$ (see Theorem 4.3 in [11]).

We define a matrix norm in terms of a given vector norm $|\cdot|$ in $\mathbb{R}^n,(n\in\mathbb{N})$, noted also $|\cdot|$. We associate with this matrix norm the Lozinskĭ measure [16], defined by

$$\mu(B)=\lim_{h\to 0^+}\frac{|I+hB|-1}{h},$$

where I denotes the unit matrix. We prove the following theorem, which gives our main result.

Theorem 5.2. *Suppose the hypothesis* (H_0) *and* (H_1) *hold.*
If $R_0>1$, then the unique positive equilibrium P is globally asymptotically stable in T.

Proof. To prove this theorem, it suffices to choose a suitable vector norm $\mathbb{R}^3$ and a matrix $A(x)$ such that

$$\bar{q}_2 := \lim_{t\to\infty} \sup \sup_{x_0 \in K} \frac{1}{t}\int_0^t \mu_1 (B(x(s,x_0)ds < 0 \tag{5.1}$$

where μ is the Lozinskĭ measure, $x=(S,E,I), B = A_g A^{-1} + AJ^{[2]}A^{-1}$, and $g(x)$ denote the vector field of (1.5). For this, we calculate as following the Jacobian matrix J of system (1.5) at x, and its second additive compound matrix $J^{[2]}$:

$$J = \begin{pmatrix} -\mu - m'(U)V & \beta & -m(U) \\ m'(U)V & -(\alpha+\beta) & m(U) \\ 0 & \sigma & -\delta \end{pmatrix}$$

$$J^{[2]} = \begin{pmatrix} -\mu - m'(U)V - (\alpha+\beta) & m(U) & f(U) \\ \sigma & -\mu - m'(U)V - \delta & 0 \\ 0 & m'(U)V & -(\alpha+\beta) \end{pmatrix}$$

So for $A(x) = diag\left\{1, \frac{E}{V}, \frac{E}{V}\right\}$, we get

$$A_g A^{-1} = diag\left\{0, \frac{E'}{E} - \frac{V'}{V}, \frac{E'}{E} - \frac{V'}{V}\right\},$$

and

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{5.2}$$

where

$$B_{11} = -\mu - m'(U)V - (\alpha+\beta), \qquad B_{12} = \frac{V}{E}(m(U) \quad m(U)), \qquad B_{21} = \begin{pmatrix} \frac{\sigma E}{V} \\ 0 \end{pmatrix},$$

$$B_{22} = \begin{pmatrix} -\mu - m'(U)V + \frac{E'}{E} - \frac{V'}{V} & 0 \\ m'(U)V & -\alpha - \delta + \frac{E'}{E} - \frac{V'}{V} \end{pmatrix}.$$

Let $\bar{\mu}_1$ denote the Lozinskiĭ measure with respect to the norm $(u,v,w) \in \mathbb{R}^3 \mapsto |(u,v,w)| = \max\{|u|, |v|+|w|\}$. Using the method of in ([14]), we get

$$\bar{\mu}_1(B) \le \sup\{g_1, g_2\} \tag{5.3}$$

where

$$g_1 = \mu_1(B_{11}) + |B_{12}|$$

$$g_2 = \mu_1(B_{22}) + |B_{21}|.$$

Thus

$$\mu_1(B_{11}) = -\mu - f'(U)V - (\alpha+\beta), \tag{5.4}$$

$$|B_{12}| = m(U)\frac{V}{E},$$

$$|B_{21}| = \frac{\sigma E}{V}.$$

Moreover, by the method in [15], we obtain

$$\mu_1(B_{22}) = \frac{E'}{E} - \frac{V'}{V} - \delta - \min\{\mu, \alpha\}, \tag{5.5}$$

$$g_1 = -\mu - m'(U)V + m(U)\frac{V}{E} - (\alpha + \beta), \tag{5.6}$$

$$g_2 = \frac{\sigma E}{V} + \frac{E'}{E} - \frac{V'}{V} - \delta - \min\{\mu, \alpha\}. \tag{5.7}$$

Therefore, we obtain respectively,

$$\frac{E'}{E} = \frac{m(U)V}{E} - (\alpha + \beta), \tag{5.8}$$

$$\frac{V'}{V} = \frac{\sigma E}{V} - \delta. \tag{5.9}$$

Substituting (5.8) into (5.6) and (5.9) into (5.7), respectively, we have

$$g_1 \le \frac{E'}{E} - \mu, \tag{5.10}$$

$$g_2 \le \frac{E'}{E} - \min\{\mu, \alpha\}. \tag{5.11}$$

Let $\eta = \min\{\mu, \alpha\}$. Since (1.5) is uniformly persistent when $R_0 > 1$, there exists $c > 0$ and $t_0 > 0$ such that $t > t_0$ implies $c \le E(t) \le \frac{\Lambda}{d}$ and $c \le V(t) \le \frac{\Lambda}{d}$ for all $(U(0), E(0), V(0)) \in K$. For $t > t_0$ we have

$$\frac{1}{t}\int_0^t \overline{\mu}_1(B)ds \le \frac{1}{t}\int_0^{t_0} \overline{\mu}_1(B)ds + \frac{1}{t}\log\frac{E(t)}{E(t_0)} - \eta\frac{t - t_0}{t} \le \frac{-\eta}{2},$$

for all $(U(0), E(0), V(0)) \in K$, which implies $\overline{q}_2 < 0$. This concludes the proof.

6. Numerical Examples

In this section, we consider the following saturated matching function: $m(U) = \frac{kU}{1 + \kappa U}$. Here, kU calculates the matching strength between vacancies and unemployed people, and $\frac{1}{1 + \kappa U}$ calculates the inhibitory effect of crowding of unemployed people when the number of individuals unable to occupy the vacancies due to the inadequacy of the training to the needs of the labor market. The following simulations investigate two cases:

- **Case 1:** $R_0 \le 1$. In this case, we fix the parameters of our system as follows:
  For these values, we find $R_0 = 0.97$, and the system (1.5) admits a unique nontrivial equilibrium P_0 which is globally asymptotically stable (see Figure 1). This result shoz that the uneployment-free eauilibrium P_0 will be reduced to very low percentages.

Table 1: Values of Parameters

Parameter	Λ	k	κ	δ	α	μ	β	σ
Value	10	0.1	3	0.02	0.05	0.014	0.8	0.5

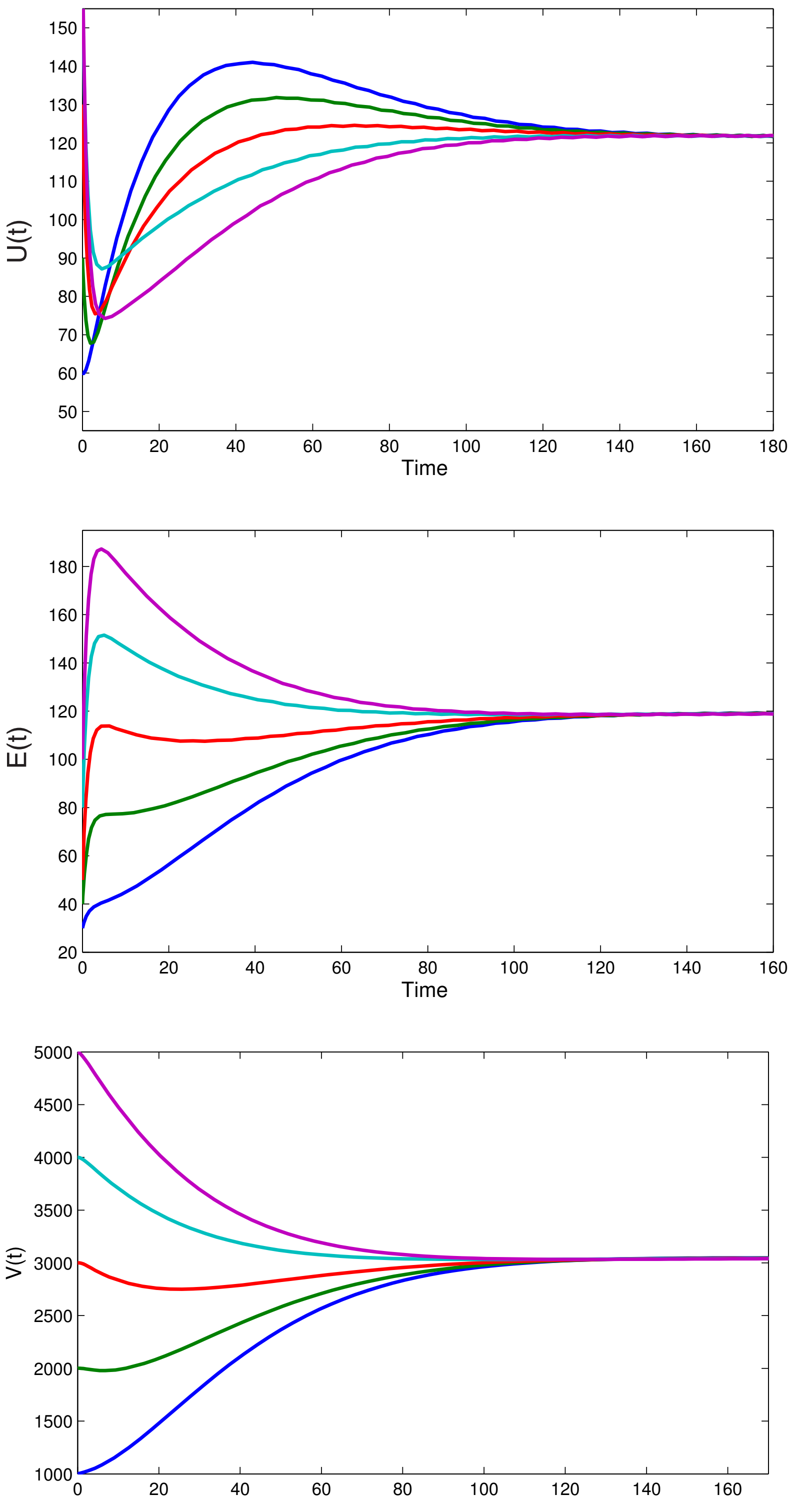

Figure 1: The trivial equilibrium, P_0, is globally asymptotically stable for ($R_0 < 1$ and $\frac{\alpha}{\beta} = 0.0625$).

- **Case 2:** $R_0 > 1$. In this case, we simulate model (1.5)with the following parameter values: We obtain $R_0 = 1.51$ and show the appearance of the second positive equilibrium, P. This equilibrium is globally asymptotically stable for two values of the rate of resigned or fired for jobs: $\beta = 0.5$ (see, Figure 2.) and $\beta = 0.8$ (see, Figure 3). This result shows that the global asymptotic stability does not depend on this parameter. Here, β satisfies neither the ElFadily condition [13] (β close to zero) nor the Maalwi condition [8] $\left(1 < R_0 < \dfrac{\alpha}{\beta}\right)$. Our result demonstrates that once the positive equilibrium airses (if $R_0 > 1$) unemployment persists in the population.

Table 2: Values of Parameters

Parameter	Λ	k	κ	δ	α	μ	σ
Value	10	0.1	3	0.02	0.05	0.014	0.5

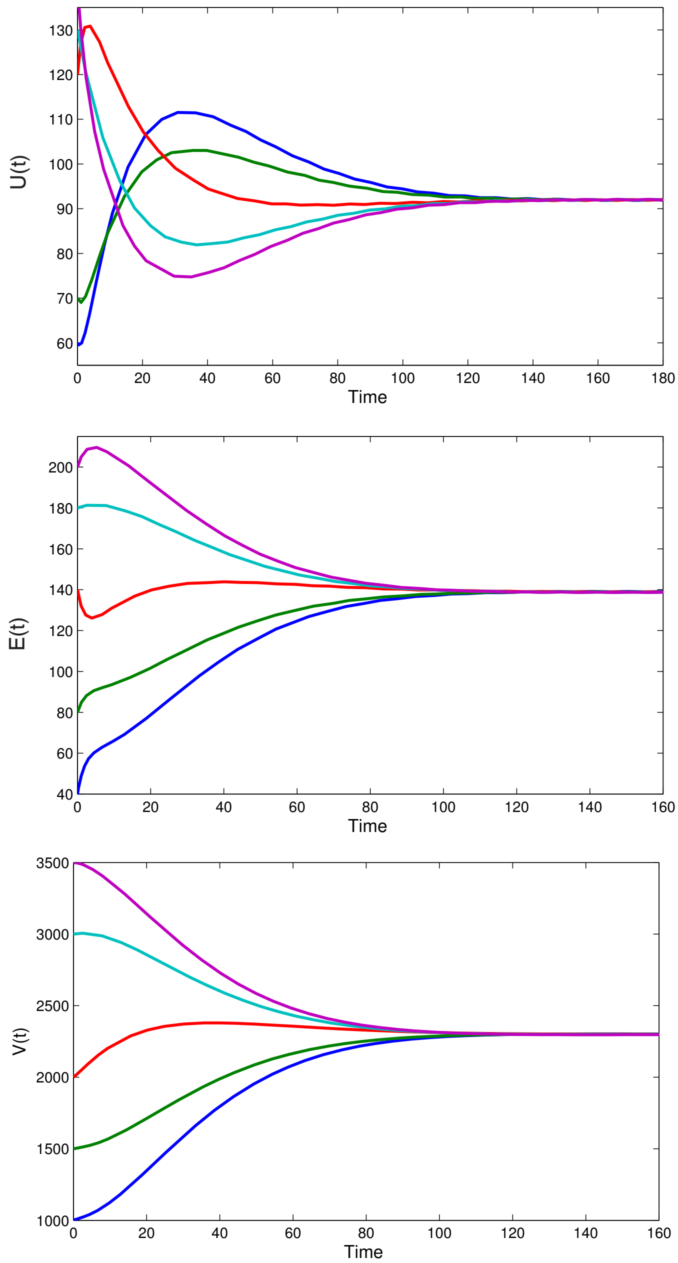

Figure 2: The positive equilibrium, P, is globally asymptotically stable for $\beta = 0.5$ ($R_0 > 1.51$ and $\dfrac{\alpha}{\beta} = 0.1$).

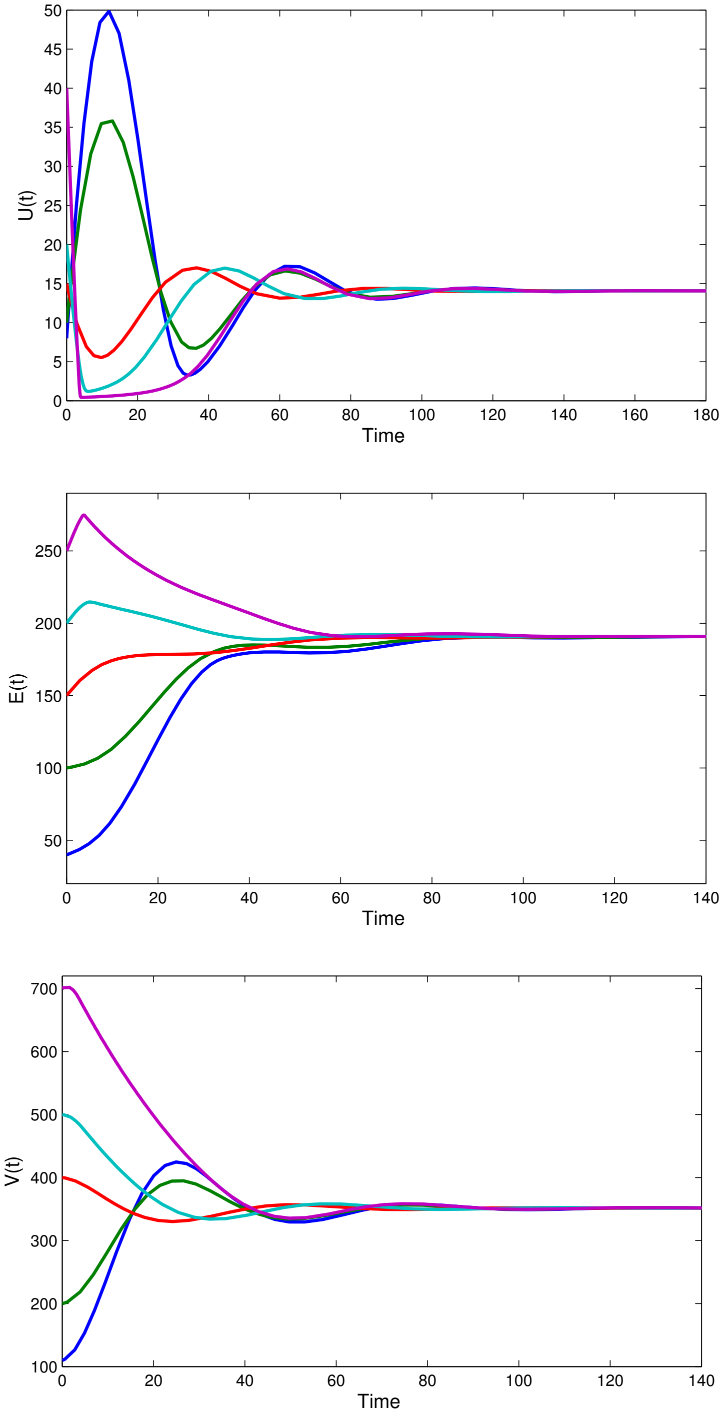

Figure 3: The positive equilibrium, P, is globally asymptotically stable for $\beta = 0.01$ ($R_0 = 13.5754$ and $\frac{\alpha}{\beta} = 5$).

7. Conclusion

In [8], Maalwi et al. studied the model (1.5), under the two hypotheses: The first one assumes that the rate of new job creation is equal to the rate of jobs loss by migration, retirment or death ($\alpha = \sigma$). The second supposes that the function modeling the matching process, between vacancies and unemployed, is linear ($m(U) = kU$). The author used a Lypunov function to prove that the non-trivial equilibrium is globally asymptotically stable, provided that the basic reproduction number R_0 is bounded between 1 and $\frac{1+\alpha}{\beta}$.

This paper presents a mathematical study of the global asymptotic behavior of an unemployment model with a general matching function and $\alpha \neq \sigma$. Which leads to find the following basic reproduction number $R_0 = \frac{\sigma m\left(\frac{A}{\mu}\right)}{\delta(\alpha+\beta)}$. This number is a key parameter for the existence and stability of positive equilibrium because it determines whether Unemployment will drop out or persist in the population as time increases. Using Lyapunov's method, we have shown that if $R_0 \leq 1$, the unemployment-free equilibrium P_0 is globally asymptotically stable, which means that the chomage cannot persist in the population, and the situation is under control. Moreover, we have proved that if $R_0 > 1$, the unemployment-free equilibrium becomes unstable and the model has a positive equilibrium *P*. To our knowledge, the study of the asymptotic stability by the method of Lyapunov gave a partial result: *P* is globally asymptotically stable provided that $1 < R_0 < 1 + \frac{\alpha}{\beta}$ (see [8]) or $R_0 > 1$ and β is close enough to zero (see [13]). Using the classical geometric approach to stability, we proved that *P* is globally asymptotically stable, without any additionnal condition. This result show that the unemployment, if initially present, will be persistent at the unique positive equilibrium level. In the end, some numerical simulations are presented to illustrate the feasibility of the theoretical results. We would like to point out here that our result, on the global stability of the unemployment-free equilibrium P_0, can be extended to the associted fractional model to our system (For more details see [17–19]). However, the global stability of the positive equilibrium *P*, of the fractional model, has not yet been tackled. We leave this for future work.

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