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Cobweb model with two delays: Stability and bifurcation

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Abstract

In this work, we study the fluctuations in the prices of agricultural and energy products. These prices are characterized by seasonality, where demand and supply conditions alternate cyclically-with a precise and known periodicity. We investigate the dynamics of a cobweb model, consisting of a one-dimensional differential equation with two-time delays, to understand the impact of demand and supply parameters on this cyclical behavior. We start by studying the linear stability analysis to find sufficient conditions under which the positive equilibrium is locally asymptotically stable. After choosing the delays as bifurcation parameters, we prove the existence of a family of periodic solutions that bifurcate from this equilibrium. Finally, we discuss their economic rationale with the help of numerical simulations.

Keywords: Cobweb model, Stability, Delay differential equations, Hopf bifurcation.

1. Introduction

The Cobweb model was introduced to provide a theoretical explanation for temporary fluctuations in market equilibrium with a supply delay. This model typically describes markets for perishable products, in which production plans must be determined before the price that will prevail in the market is known. Thus, producers must form expectations about future prices and must base their production decisions on these expectations. In this way, the selling price is primarily influenced by the price

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forecast, the resulting production decisions, and the anticipated future demand (which participates in the evaluation of the price forecast) rather than by current demand. This model was originally developed by Kaldor [1] with linear supply and demand. Subsequently, thanks to some new mathematical results as well as the dramatic increase in computing capacity, nonlinear versions of the Cobweb model have been studied and carried out through simulation experiments, showing persistent oscillations in prices [2, 3]. A further step in this line of research was to include more realistic mechanisms with regard to expectations formations in the Cobweb framework. For this purpose, Chiarella [4] and Hommes [5, 6] have studied the dynamics of prices with nonlinear supply and demand curves by using the adaptive expectations hypothesis [7], with respect to which prices are revised according to prediction errors of agents.

Along the same line of research, Onozaki et al. [8] have revisited the cobweb model by considering adaptive adjustments on the quantity produced instead of price expectations. With this behavioral rule, farmers partially adjust production in the direction of the best response (represented by the quantity that maximizes expected profits). By assuming a nonlinear (monotonic) market demand, they showed, by using the Homoclinic Point Theorem, that topological chaos can occur in a model whose dynamics are characterized by a one-dimensional map. Mackey [9] gives a nonlinear price adjustment model with production delay and rigorously derives a stability switching condition for which the stability of equilibrium is lost. Furthermore, it is shown that a Hopf bifurcation takes place and thus the stable equilibrium bifurcates to a limit cycle after the loss of stability. Recently, Gori et al. [10] proposed a delay Cobweb model with profit-maximizing behavior to characterize production cycles. The dynamics of the economy are characterized by a one-dimensional delay differential equation. Moreover, they take, in [11], a dynamic view of quantities rather than prices by also assuming heterogeneous interacting agents, thus allowing the economy to be described by a system of two delay differential equations instead of a one-dimensional system. In this context, by applying the recent techniques developed by Ruan and Wei [12] and the geometric approach of stability crossing curves developed by Gu et al. [13], they showed the role of heterogeneity in the emergence of Hopf bifurcations.

The Cobweb model has also been studied using the gradient approach in [3]. The authors have shown that the introduction of the gradient mechanism in the price adjustment process does not always lead to market equilibrium, but can also induce complex dynamics and endogenous fluctuations in price evolution. This mechanism is also used, by Askar, to study two discrete-time dynamic systems: one-dimensional (1D) and two-dimensional (2D). The first model has three equilibrium points, but only the stability of the non-zero real price equilibrium point is examined. In the second, where the memory factor is introduced, the equilibrium price can be destabilized by chaotic behavior, which is formed due to period doubling and the Neimark-Sacker bifurcation.

Also contains other works which have suggested that producers may not be rational [14], others work purpose a version of the Cobweb model with adaptive expectations, accordingly modified to be consistent with the market's seasonality (see the work of Cavalli et al. [15]). The aim was to understand how the periodic nature of the market, as well as the mechanism of formation of agents' expectations, affect the resulting dynamics.

Although the models, with multiple delays, have been an object of study for a long time, the Cobweb models are subject to only one-time delay and little is known about multiple delay versions. In this way, we cite the work of Matsumoto [16], who studied the stability conditions of a Cobweb model with two delays.

This document is organized as follows. Section 2 presents the basic model of Cobweb and its version with two delays. In section 3, we give the stability analysis of the positive equilibrium and the existence of a Hopf bifurcation around this equilibrium. Numerical simulations are presented in section 4, and the article is concluded in section 5.

2. The Model

2.1 Basic Cobweb Model

The basic formulation of the Cobweb model considers a situation in a single-good economic system, where the demand and supply functions for a given good are respectively given by

$$D(p(t)) = d_1 - d_2 p(t) \text{ and } S(p^e(t)) = s_1 + s_2 p^e(t),$$

with $p^e = p(t-1)$. Traditionally, it's supposed that demand negatively depends on price while supply positively depends on the expected price. For the sake of analytical simplicity, it's also assumed that consumers and producers make their decisions based only on the price information appearing in the good market.

The constants d_1, d_2, s_1, and s_2 are all taken to be positive, with d_2 representing the price sensitivity of demand at time t and s_2 representing the price sensitivity of supply at time $t - 1$, respectively. In the supply function equation, the quantity to be supplied by the producer has been expressed in terms of the price at time $t - 1$. Such situations occur in economics. Suppose that in each period the market price is always set at a level such that the quantity demanded equals the quantity supplied. In other words,

$$S = D.$$

The price change is:

$$\frac{dp(t)}{dt} = \kappa p(t)[D(p(t)) - S(p^e(t))], \tag{2.1}$$

where κ is the adjustment coefficient.

2.2 Cobweb Model with Two Delays

The purpose of this study is, based on Matsumoto's [16] formulation, to consider the expected price depending on two prices $t-\tau_1$ and $t-\tau_1$;

$$p^e(t) = \theta p(t-\tau_1) + (1-\theta)p(t-\tau_2), \tag{2.2}$$

where $p(t-\tau_1)$ and $p(t-\tau_2)$ respectively are the price realized at time $t-\tau_1, t-\tau_2$, and θ is a weight parameter $0<\theta<1$.

Accordingly the price adjustment is governed by a two-delay differential equation

$$\frac{dp}{dt} = \kappa p(t)[(d_1 - s_1) - (d_2 p(t) - s_2\theta p(t-\tau_1) - s_2(1-\theta)p(t-\tau_2))]. \tag{2.3}$$

The study of the dynamics of the (2.3) equation has a long history. A great effort has been devoted to the analysis of the local stability and the existence of Hopf bifurcation. For instance, Braddock et al [15] have completely examined the stability of the equation (2.3). The discussion of the existence of the Hopf bifurcation of the equation (2.3) also appears in the works [12, 17, 18] in the case where $\kappa d_2 = 0$. The analysis of the Hopf bifurcation in the case where $\kappa d_2 \neq 0$, is very complex and remains an open question. In our paper, we focus on this question and look for the sufficient condition under which a family of periodic solutions bifurcates from the positive equilibrium.

3. Positive Equilibrium and its Stability Analysis

3.1 Positive Equilibrium

In the following proposition, we prove the existence of the unique positive equilibria.

Proposition 3.1. *The model (2.1) has two equilibria : the trivial equilibrium* $p = 0$*, for any values of parameters and the positive equilibria* $p^* = \dfrac{d_1 - s_1}{d_2 + s_2}$.

Remark 3.2. *The assuption* $d_1 > s_1$ *is natural, because the minimum demand is always excded by the maximum supply.*

3.2 Local Stability and Hopf Bifurcation Analysis

By analysing the characteristic equation associated to (2.3), we prove the existence of Hopf bifurcation around the equilibrium p^*.

The characteristic equation corresponding to the model (2.3) arround the equilibrium p^* takes the form:

$$\nu = -\Lambda_0 - \Lambda_1 e^{-\nu\tau_1} - \Lambda_2 e^{-\nu\tau_2}, \tag{3.1}$$

where $\Lambda_0 = \kappa p^* d_2, \Lambda_1 = \kappa p^* \theta s_2$ and $\Lambda_2 = \kappa p^*(1-\theta)s_2$. Lets $\lambda = \dfrac{\nu}{\Lambda_1}, A = \dfrac{\Lambda_2}{\Lambda_1}, B = \dfrac{\Lambda_0}{\Lambda_1}, \sigma_1 = \Lambda_1\tau_1$, and $\sigma_2 = \Lambda_1\tau_2$. We obtain the normalized characteristic equation:

$$\lambda = -B - e^{-\lambda\sigma_1} - Ae^{-\lambda\sigma_2}. \tag{3.2}$$

Hence, according to the Gopalsamy work [10] (see also [15]), we have the following lemma.

Lemma 3.3. *For* $\sigma_2 = 0$

- If $(1-2\theta)s_2 + d_2 > 0$, then all roots of the characteristic equation (3.2) have negative real parts;
- If $(1-2\theta)s_2 + d_2 < 0$, then there exists an σ_1^* such that if $\sigma_1 < \sigma_1^*$, then all roots of the characteristic equation (3.2) have negative real parts and if $\sigma_1 > \sigma_1^*$, then the characteristic equation (3.2) has at least one root with positive real part with

$$\sigma_1^* = \frac{arcos(-A-B)}{\sqrt{1-(A+B)^2}}.$$

Proof. For $\sigma_2 = 0$, the normalized characteristic equation (3.2) reads as

$$\lambda = -(A+B) - e^{-\lambda\sigma_1}. \tag{3.3}$$

- One just needs to see that $(1-2\theta)s_2 + d_2 > 0$, if and only if $A + B > 1$;
- Simply needs to see that $(1-2\theta)s_2 + d_2 < 0$, if and only if $A + B < 1$.

Remark 3.4. *It's easy to verify that* $A + B > 0$.
Let's now return to the study of equation (3.2) when $\sigma_2 \geq 0$.

Lemma 3.5. *Assume that*

$$(H_0): d_2 > s_2.$$

Then all roots of the characteristic equation (3.2) have negative real parts for any $\sigma_i \geq 0, i = 1;2$.

Proof. If (H_0) holds, then we get $(1-2\theta)s_2 + d_2 > 0$, and from (i) in lemma (3.3) all roots of the characteristic equation (3.2) have negative real parts for $\sigma_2 = 0$, and for $\sigma_2 \geq 0$. If this conclusion fails, then there must be some σ_2^* such that equation (3.2) has purly imaginary roots $\omega, (i\omega > 0)$ satisfying:

$$\begin{aligned} &B + \cos(\omega\sigma_1) = -A\cos(\omega\sigma_2) \\ &\omega - \sin(\omega\sigma_1) = \sin(\omega\sigma_2). \end{aligned} \tag{3.4}$$

Adding up the squares of both equations, we have

$$\omega^2 - 2\omega\sin(\omega\sigma_1) + 2B\cos(\omega\sigma_1) + B^2 + 1 = A^2,$$

that is

$$\omega^2 - 2\omega\sin(\omega\sigma_1) + \sin^2(\omega\sigma_1) + \cos^2(\omega\sigma_1) + 2B\cos(\omega\sigma_1) + B^2 + 1 = A^2,$$

so

$$(\omega - \sin(\omega\sigma_1))^2 + (\cos(\omega\sigma_1) + B)^2 = A^2.$$

Then we can let's

$$\begin{aligned} &g_1(\omega) = (\omega - \sin(\omega\sigma_1))^2; \\ &g_2(\omega) = A^2 - (\cos(\omega\sigma_1) + B)^2. \end{aligned} \tag{3.5}$$

The functions g_1 and g_2 have the following propreties:

1. $g_1(\omega) \geq 0, \quad g_1(0) = 0$ and $\lim_{\omega\to+\infty} g_1(\omega) = +\infty$;
2. If $B > 1$, then $A^2 - (B+1)^2 \leq g_2(\omega) \leq A^2 - (B-1)^2$;
3. If $B \leq 1$, then $0 \leq g_2(\omega) \leq A^2 - (B+1)^2$.

When $(H0)$ holds, then we get $A^2 - (B-1)^2 < 0$. Consequntly, we can see that

$$g_2(\omega) < 0 \leq g_1(\omega). \tag{3.6}$$

Thus, the equation $g_1(\omega) = g_2(\omega)$ has no solution, and consequently all roots of equation (3.2) have strictly negative real parts for all time delays $\sigma_i \geq 0,\ i = 1;2$.

Lemma 3.6. *Suppose that the following assertions hold:*

$$\begin{aligned} &(H_1): (1-2\theta)s_2 + d_2 < 0 \\ &(H_2): s_2 < -\theta d_2 + \frac{\sqrt{\theta^2 d_2^2 + ((1-\theta)^2 d_2^2}}{(1-\theta)^2}. \end{aligned}$$

Then all roots of the characteristic equation (3.2) have negative real parts for $\sigma_1 < \sigma_1^*$ and $\sigma_2 \geq 0$, where σ_1^* is defined in Lemma (3.3).

Proof. For $\sigma_1 < \sigma_1^*$, if equation (3.2) has purly imaginary roots $\pm i\omega$ with $(\omega > 0)$, we can write:

$$\begin{aligned} &B + \cos(\omega\sigma_1) = -A\cos(\omega\sigma_2), \\ &\omega - \sin(\omega\sigma_1) = \sin(\omega\sigma_2). \end{aligned} \tag{3.7}$$

Adding up the squares of both equations, we have

$$\omega^2 - 2\omega\sin(\omega\sigma_1) + 2B\cos(\omega\sigma_1) + B^2 + 1 = A^2.$$

So

$$\frac{\omega^2 + 2B\cos(\omega\sigma_1) + B^2 + 1 - A^2}{2\omega} = \sin(\omega\sigma_1). \tag{3.8}$$

If we take

$$h(\omega)=\frac{\omega^2+2B\cos(\omega\sigma_1)+B^2+1-A^2}{2\omega}; \tag{3.9}$$

we obtain

$$h(\omega)=\sin(\omega\sigma_1). \tag{3.10}$$

Since $|\cos(\omega\sigma_1)|\le 1$, it follows that

$$h_1(\omega)\le h(\omega)\le h_2(\omega)), \tag{3.11}$$

with

$$h_1(\omega)=\frac{\omega^2+(B-1)^2+1-A^2}{2\omega}, \tag{3.12}$$

and

$$h_2(\omega)=\frac{\omega^2+(B+1)^2+1-A^2}{2\omega}. \tag{3.13}$$

it's easy to verify that $(1-2\theta)s_2+d_2<0$ if and only if $A+B<1$, and $s_2<\dfrac{-\theta d_2+\sqrt{\theta^2 d_2^2+(1-\theta)^2 d_2^2}}{(1-\theta)^2}$ implies that $(B-1)^2+1-A^2>1$. Furthermore, for $A+B<1$ and $(B-1)^2+1-A^2>1$, the function h_1 defined by (3.12) has the following properties (for the case $B=0$, we get the result similar to that found in [19]):

1. h_1 attains its unique minimum when $\omega=\sqrt{(B-1)^2+1-A^2}$;
2. $\lim_{\omega\to 0^+}h_1(\omega)=+\infty$ and $\lim_{\omega\to+\infty}h_1(\omega)=+\infty$;
3. For $\sigma_2>\sigma_2^*$, the characteristic equation (3.2) has at least one root with positive real part.

Clearly, for $A+B<1$ and $(B-1)^2+1-A^2>1$, the function h_1 and $\sin(\omega\sigma_1)$ do not intersect. Consequently, all roots of the characteristic equation (3.2) have negative real parts for all $\sigma_1<\sigma_1^*$, and $\sigma_2\ge 0$.

Lemma 3.7. *Suppose that the following assertion hold:*

$$(H_3):(2\theta-1)s_2<d_2<(1-2\theta)s_2.$$

Then there exist $\sigma_2^*>0$ such that:

- If $\sigma_2<\sigma_2^*$, then all roots of the characteristic equation (3.2) have negative real parts;
- If $\sigma_2=\sigma_2^*$, then the characteristic equation (3.2) has a pair of purly imaginary root $\pm i\omega_0$;
- If $\sigma_2>\sigma_2^*$, then the characteristic equation (3.2) has at least one root with positive real part.

Proof. We use the same technique employed in the proof of Lemma (3.6). It's easy to see that $(2\theta-1)s_2<d_2<(1-2\theta)s_2$ implies $(B-1)^2+1-A^2<0$ and $(B+1)^2+1-A^2<0$. Furthermore, for $(B-1)^2+1-A^2<0$, and $(B+1)^2+1-A^2<0$, the function h_1 and h_2, defined respectively by (3.12) and (3.13) has the following properties (for the case $B=0$), we get the result similar to that found in [19]):

1. h_1 and h_2 are concave and strictly monotonically increasing;
2. $\lim_{\omega\to 0^+}h_1(\omega)=\lim_{\omega\to 0^+}h_2(\omega)=-\infty$;

3. $\lim_{\omega\to+\infty} h_1(\omega) = \lim_{\omega\to+\infty} h_2(\omega) = +\infty$;
4. $h_1(\beta) = 1$ and $h_1(\alpha) = -1$,

where $\alpha = \sqrt{1 + A^2 - (B+1)^2 - 1}$ and $\beta = \sqrt{1 + A^2 - (B+1)^2 + 1}$.

Clearly, for every $\sigma_1 \geq 0$ the function $h(\omega)$, defined by (3.9), intersects $\sin(\omega\sigma_1)$ only in the rectangle bounded by $y = \pm 1, \omega = \alpha$ and $\omega = \beta$; that means, if equation (3.2) has a pair of purly imaginary root $\pm i\omega_0$ $(\omega > 0)$, then $\omega \in]\alpha;\beta[$.

Let $\omega_1, \omega_1, \ldots, \omega_m$, $(m \geq 1)$ the solutions of equation (3.10). It follows from $A + B > 1$ that, for all $j \in \{1, 2, \ldots, m\}$ we have

$$\sigma_{2j} = \frac{1}{\omega_j} \arccos\left(\frac{-B - \cos(\omega_j\sigma_1)}{A}\right)$$

where (ω_j, σ_{2j}) is the solution of the system (3.4) which satisfies $\omega_j\sigma_{2j} \in]0;\pi[$.

Let

$$\sigma_2^* = \min\{\sigma_{2j}\},\ j = 1, \ldots, m,$$

and $\omega_0 = \omega_j$ such that

$$\sigma_2^* = \frac{1}{\omega_0} \arccos\left(\frac{-B - \cos(\omega_0\sigma_1)}{A}\right). \tag{3.14}$$

Thus, we have:

1. If $\sigma_2 < \sigma_2^*$, then all roots of the characteristic equation (3.2) have negative real parts;
2. If $\sigma_2 = \sigma_2^*$, then the characteristic equation (3.2) has a pair of purly imaginary root $\pm i\omega_0$;
3. If $\sigma_2 > \sigma_2^*$, then the characteristic equation (3.2) has at least one root with positive real part.

Lemma 3.8. *Suppose that the following assertion hold:*

$$(H_4): (1 - 2\theta)s_2 < d_2 < (2\theta - 1)s_2,$$

then there exist $\sigma_1^* > 0$ and $\sigma_2^* > 0$ such that

1. If $\sigma_1 < \sigma_1^*$, and $\sigma_2 < \sigma_2^*$, then all roots of the characteristic equation (3.2) have negative real parts;
2. If $\sigma_1 < \sigma_1^*$, and $\sigma_2 = \sigma_2^*$, then the characteristic equation (3.2) has a pair of purly imaginary roots $\pm i\omega_0$;
3. If $\sigma_1 < \sigma_1^*$, and $\sigma_2 > \sigma_2^*$, then the characteristic equation (3.2) has at least one root with positive real part.

Proof. From (ii) in Lemma (3.3), if $d_2 < (2\theta - 1)s_2$ and $\sigma_2 = 0$, then there exists an $\sigma_1^* > 0$ such that all roots of the characteristic equation (3.2) have negative real parts, for all $\sigma_1 < \sigma_1^*$. If this conclusion fails, then there must be some $\sigma_2^* > 0$ such that equation (3.2) has purly imaginary roots $\pm i\omega$, where ω is a positive solution of the following equation:

$$h(\omega) = \sin(\omega\sigma_1),$$

where

$$h(\omega) = \frac{\omega^2 + 2B\cos(\omega\sigma_1) + B^2 + 1 - A^2}{2\omega}.$$

Since $|\cos(\omega\sigma_1)| \le 1$, it follows that

$$h_1(\omega) \le h(\omega) \le h_2(\omega),$$

with h_1 and h_2 are defined respectively by (3.12) and (3.13).

It is easy to veriy that $(1-2\theta)s_2 < d_2 < (2\theta-1)s_2$ implies that $(B-1)^2 - A^2 < 0$ and $(B+1)^2 - A^2 < 0$. The proof for the rest of the Lemma is similar to the proof of Lemma (3.7).

Collecting together all this information about characteristic roots of equation (3.2) we have the following results.

Theorem 3.9. *Assume that* (H_0) *holds. Then the positive equilibruim of equation (3.1) is locally asymptotically stabe for any* $\tau_i \ge 0,\ i = 1;2$.

Proof. From Lemma (3.5), the characteristic equation (3.2) have negative real parts for any $\tau_i \ge 0, i = 1;2$. Using de variable changes $\sigma_i = \Lambda_i \tau_i, i = 1;2$, the results remains true for characteristic equation (3.1), for any $\tau_i \ge 0, i = 1;2$. Thus, the positive equilibruim of equation (3.1) is locally asymptotically stabe.

Theorem 3.10. *Suppose that the following assertions* (H_1) *and* (H_2) *hold. Then there exists an* $\tau_1^* = \dfrac{\arccos(-A-B)}{\Lambda_1\sqrt{1-(A+B)^2}}$ *such that if* $\tau_1 < \tau_1^*$ *the positive equilibruim of equation* (3.1) *is locally asymptotically stabe for any* $\tau_2 \ge 0$.

Proof. From Lemma (3.6), if $\tau_2 \ge 0$, and $\tau_1 \in [0, \tau_1^*)$, then all roots of the characteristic equation (3.2) have negative real parts. Using de variable changes $\sigma_i = \Lambda_i \tau_i,\ i = 1;2$, the results remains true for characteristic equation (5), for any $\tau_1 \in [0, \tau_1^*)$ and any $\tau_2 \ge 0$, with $\tau_1^* = \dfrac{\arccos(-A-B)}{\Lambda_1\sqrt{1-(A+B)^2}}$. Consequently, the positive equilibruim of equation (3.1) is locally asymptotically stabe for any $\tau_2 \ge 0$, and $\tau_1 \in [0, \tau_1^*)$.

Theorem 3.11. *Suppose that* (H_3) *holds. Then there exists* $\tau_2^* = \dfrac{\sigma_2^*}{\Lambda_2}$ *such that, the positive equilibruim of equation (3.2) is locally asymptotically stabe for any* $\tau_1 \ge 0$, *and* $\tau_2 < \tau_2^*$; *where* τ_2^* *is defined by (3.14).*

Proof. From Lemma (3.7), there exists τ_2^* such that, if $\tau_1 \ge 0$ and $\tau_2 \in [0, \tau_2^*)$, then all roots of the characteristic equation (3.2) have negative real parts. Using de variable changes $\sigma_i = \Lambda_i \tau_i,\ i = 1;2$, the results remains true for characteristic equation (3.1), for any $\tau_1 \ge 0$, and any $\tau_2 \in [0, \tau_2^*)$ with $\tau_2^* = \dfrac{\sigma_2^*}{\Lambda_2}$. Consequently, the positive equilibruim of equation (3.1) is locally asymptotically stabe for $\tau_1 \ge 0$ and $\tau_2 < \tau_2^*$.

Theorem 3.12. *Suppose that* (H_3) *holds and* $\tau_2 = \tau_2^*$. *If one of the following situatios holds:*

$$(H_5): \omega_0 \tau_2^* \in \left[0, \frac{\pi}{2\Lambda_2}\right)$$

$$(H_6): \omega_0 \tau_2^* \in \left[\frac{\pi}{2\Lambda_2}, \frac{\pi}{\Lambda_2}\right) \text{ and } \tan\left(\frac{\omega_0 \tau_2^*}{\Lambda_2}\right) < \frac{-\omega_0 \Lambda_1 \tau_1}{1+\Lambda_1 \tau_1}.$$

Then, there exists $\varepsilon_0 > 0$ such that, for each $0 \le \varepsilon < \varepsilon_0$, equation (3.1) has a family of periodic solutions $p(\varepsilon)$ with period $T = T(\varepsilon)$ for the parameter values $\tau_2 = \tau_2(\varepsilon)$, such that $p(0) = 0$, $T(0) = \dfrac{2\pi}{\omega_0}$ and $\tau(0) = \tau_2^* = \dfrac{\sigma_2^*}{\Lambda_2}$, where σ_2^* is defined by (3.14) and $\pm i\omega_0$ is the purly imaginary roots of equation (3.2) corresponding to σ_2^*.

Proof. From Lemma (3.7), there exists $\tau_2^* > 0$, such that, if $\tau_2 = \tau_2^*$, then $\pm i\omega_0$ is the purly imaginary roots of equation (3.2). Next we show that $\pm i\omega_0$ are simple roots of equation (3.2). Set

$$f(\lambda) = \lambda + B + e^{-\lambda\sigma_1} + Ae^{-\lambda\sigma_2^*},$$

we have

$$\frac{df(\lambda)}{d\lambda} = 1 - \sigma_1 e^{-\lambda\sigma_1} - A\sigma_2^{*-\lambda\sigma_2^*},$$

and

$$\frac{df(i\omega_0)}{d\lambda} = 1 - \sigma_1(\cos(-\omega_0\sigma_1) - i\sin(-\omega_0\sigma_1)) - A\sigma_2^*(\cos(-\omega_0\sigma_2^*) - i\sin(-\omega_0\sigma_2^*)).$$

It's follows from (3.4) that

$$\frac{d(Re(f(i\omega_0))}{d\lambda} = 1 + \sigma_1 B + A(\sigma_1 - \sigma_2^*)\cos(\omega_0\sigma_2^*),$$
$$\frac{d(Im(f(i\omega_0))}{d\lambda} = \omega_0\sigma_1 - A(\sigma_1 - \sigma_2^*)\sin(\omega_0\sigma_2^*).$$

- If $\omega_0\sigma_2^* \in \left(0; \frac{\pi}{2}\right]$, then $\frac{d(Re(f(i\omega_0))}{d\lambda} > 0$ or $\frac{d(Im(f(i\omega_0))}{d\lambda} > 0$;
- If $\omega_0\sigma_2^* \in \left(\frac{\pi}{2}; \pi\right]$, we obtain

$$\begin{aligned} \frac{d(Re(f(i\omega_0))}{d\lambda} &= 0; \\ \frac{d(Im(f(i\omega_0))}{d\lambda} &= 0, \end{aligned} \tag{3.15}$$

is equivalent to

$$\cos(\omega_0\sigma_2^*) = \frac{-(1+\sigma_1 B)}{A(\sigma_1 - \sigma_2^*)};$$
$$\sin(\omega_0\sigma_2^*) = \frac{\omega_0\sigma_1}{A(\sigma_1 - \sigma_2^*)},$$

which implies that

$$\tan(\omega_0\sigma_2^*) = \frac{\omega_0\sigma_1}{1+\sigma_1 B}. \tag{3.16}$$

Finaly, we verify the transversally condition. Let $\lambda(\sigma_2) = \eta(\sigma_2) + i\omega(\sigma_2)$ be the root of equation (3.2) satisfying $\eta(\sigma_2^*) = 0, \omega(\sigma_2^*) = \omega_0$. Differentiating with respect to σ_2 on both sides of equation (3.2) gives

$$\frac{d\lambda(\sigma_2^*)}{d\sigma_2} = \frac{A\lambda e^{-\lambda\sigma_2^*}}{1 - \sigma_1 e^{-\lambda\sigma_1} - A\sigma_2^* e^{-\lambda\sigma_2^*}}. \tag{3.17}$$

It follows from (3.2) that

$$\eta'(\sigma_2^*) = \frac{A\omega_0[(1+\sigma_1 B)\sin(\omega_0\sigma_2^*) + \omega_0\sigma_1\cos(\omega_0\sigma_2^*)]}{[1+\sigma_1 B + A(\sigma_1 - \sigma_2^*)\cos(\omega_0\sigma_2^*)]^2 + [\omega_0\sigma_1 + A(\sigma_1 - \sigma_2^*)\sin(\omega_0\sigma_2^*)]^2}.$$

Thus,

- If $\omega_0\sigma_2^* \in \left(0;\frac{\pi}{2}\right]$, then $\frac{df(i\omega_0)}{d\lambda} \neq 0$ and $\eta'(\sigma_2^*) > 0$;
- If $\omega_0\sigma_2^* \in \left(\frac{\pi}{2};\pi\right]$, and $\tan\left(\frac{\omega_2\sigma_2^*}{\Lambda_2}\right) < \frac{-\omega_0\Lambda_1\tau_1}{1+\Lambda_0\tau_1}$, we have $\frac{df(i\omega_0)}{d\lambda} \neq 0$. Also, we have $\eta'(\sigma_2^*) > 0$.

Using the variable changes $\sigma_i = \Lambda_i\tau_i, i = 1;2$, the results remains true for characteristic equation (3.1), for $\tau_0^* = \frac{\sigma_0^*}{\Lambda_2}$, that is, there exists $\pm\tilde{\omega}_0$ a pair of purely imagnary roots of equation (3.1), which completes the proof.

4. Numerical Examples

Our current objective is to study the dynamics of the Cobweb model (2.3) when the timing parameters vary. For this we propose the following numerical values: $\kappa = 1, \theta = 0.8, d_1 = 4, d_2 = 3, s_1 = 0.5, s_2 = 2$. We find that the Cobweb model presents an equilibrium price. This equilibrium is a balance of demand and supply factors. The numerical simulations lead to one of the following scenarios: damped oscillations, limit cycles, or unstable oscillations. In Figure 1, when the two delays are lower than some critical values the equilibrium price is locally asymptotically stable. There is a tendency for prices to return in a spiral course to this equilibrium. For critical values of the two delays, we note the birth of oscillations of constant amplitude, and also constant proper period whose value depends only on the critical values of the delays (see, Figure 2). Finally, Figure 3 shows that time delays can destabilize the equilibrium price by giving unstable oscillations.

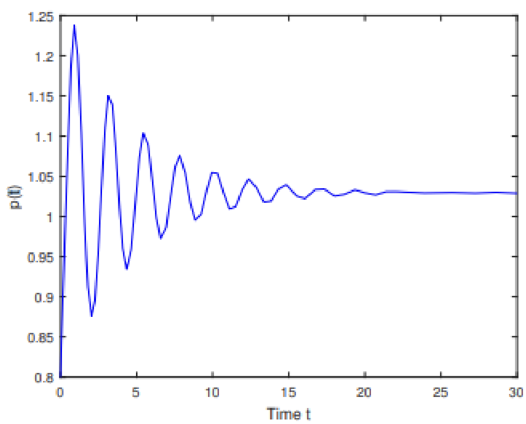

Figure 1: For $(\tau_1^*, \tau_2^*) = (0.68, 0.9)$, the equilibrium p^* of model (2.3) is locally asymptotically stable.

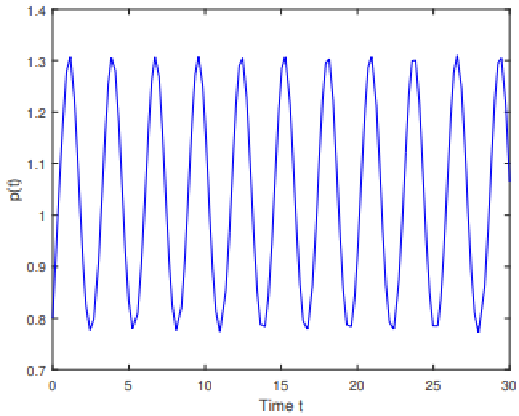

Figure 2: For $(\tau_1^*, \tau_2^*) = (0.92, 0.9)$, an oscillating movement occurs around the positive quilibrium p^*.

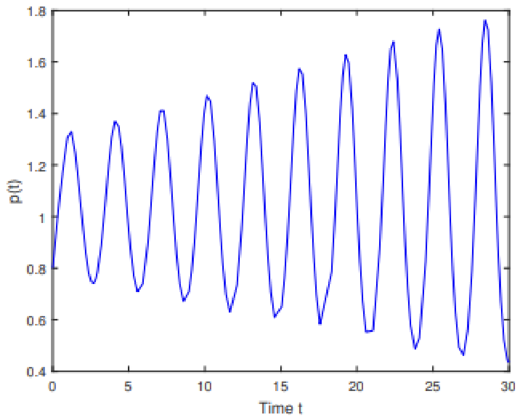

Figure 3: For $(\tau_1^*, \tau_2^*) = (0.98, 0.9)$, the equilibrium p^* of model (2.3) is unstable.

5. Conclusion

It is true that the prices of the previous periods determine the quantity produced in the next period because the price recorded in the previous period can lead to an imbalance between supply and demand (reduction of production if the price was not satisfactory or increase in the quantities produced in the opposite case) and then results in a decrease or increase of the current price in the market. In this work, we have proposed a Cobweb model with two distinct delays, τ_1 and τ_2 (indicating the time lag between the prices realized in the two previous periods $t-\tau_1$ and $t-\tau_2$ and the expected price of the period t. First, we use linear stability analysis to find sufficient conditions under which the positive equilibrium is locally asymptotically stable. Second, by choosing the two time periods as the bifurcation parameter, we prove the existence of a family of periodic solutions that bifurcate from this equilibrium. The results improve some results concerning the Hopf bifurcation problem in [12, 16]. This analysis of the bifurcation of the proposed model allowed us to conclude that the influence of the first time delay τ_1 is more important than the influence of the first time delay τ_2 because the memory (uncertainty and instability) of the majority of producers is shorter and essentially attached to the price of the previous period.

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