Results in Nonlinear Analysis 7 (2024) No. 1, 64–79
https://doi.org/10.31838/rna/2024.07.01.008
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Approximate polynomial solution for two-point fuzzy boundary value problems

Mazin H. Suhhiem[1], Raad I. Khwayyit[2]

[1]Department of Mathematics, University of Sumer, Alrifaee, Iraq; [2]Ministry of Education, Alrifaee, Iraq

Abstract

In this research, we have used double decomposition method to find approximate-analytical solutions for the two-point fuzzy boundary value problems. This method is based on the standard Adomian decomposition method, which is an approximation method that is used to solve fuzzy and non-fuzzy differential equations. This method allows for the solution to be calculated as a convergent series, this means that the solution is in the form of a polynomial that approaches the exact solution of the differential equation. The numerical solutions that we presented during this research showed the high efficiency of this method.

Key words and phrases. Two-point fuzzy boundary value problem, fuzzy approximate-analytical solution, Adomian polynomials, double decomposition method.

1. Introduction

Many methods have been developed so far for solving fuzzy differential equations (FDEs). And since the FDEs have many important applications in various types of sciences, medicine and engineering, these proposed methods included all kinds of the numerical solutions, exact-analytical solutions and approximate-analytical solutions. Finding different types of solutions gives more freedom in dealing with the FDEs, because the exact-analytical solution may be difficult or non-existent.

One of the powerful approximate-analytical methods that tackle numerous functional equations successfully: the Adomian decomposition method (ADM). ADM is a powerful decomposition

Email addresses: mazin.suhhiem@yahoo.com (Mazin H. Suhhiem)*; hgf0783@gmail.com (Raad I. Khwayyit)

Received October 19, 2023; Accepted November 21, 2023; Online December 18, 2023

methodology for the practical solution of linear or nonlinear and deterministic or stochastic operator equations, including differential equations, integral equations, etc. The method provides the solution in a rapidly convergent series with components that can be computed iteratively. Because of its high efficiency in approximating the exact-analytical solution, many students and researchers have used this method to solve FDEs (For more details, see [1, 2, 4, 6, 8– 12, 14, 16].

the main objective of this paper is to employ the double decomposition method to solve two-point fuzzy boundary value problems featuring linear and nonlinear ordinary differential equations. Recall that Adomian and Rach, in 1993, initiated the double decomposition method to improve the proficiency of the standard ADM. Further, Aminataei and Hosseini compared the double decomposition method with the standard ADM on certain boundary-value problems of the second order. Their finding was that the double decomposition method has more virtues, including higher accuracy and faster convergence, against the standard ADM (For more details, see [3, 17]).

during this work, we need many fundamental concepts in the fuzzy set theory, such as fuzzy number, fuzzy function and fuzzy derivative. These concepts can be found in detail in [5, 7, 13, 15].

2. Two-Point Fuzzy Boundary Value Problems

The general form of the two-point fuzzy boundary value problems for the ordinary differential equations is [7]:

$$u''(x) = f(x, u(x), u'(x)), x \in [a, b] \tag{2.1}$$

With:

$$u(a) = A,\ u(b) = B$$

Where:

u is a fuzzy function of the crisp variable x,

$f(x, u(x), u'(x))$ is a fuzzy function of the crisp variable x and the fuzzy variable u,

$u'(x)$ is the first order fuzzy derivative of $x, u(x)$,

$u''(x)$ is the second order fuzzy derivative of x, $u(x)$, $u'(x)$,

a and b are real numbers,

A and B are fuzzy numbers.

Equation (2.1) can be converted into [15, 16]:

$$\underline{(u''(x))} = \underline{f}(x, u, u') = H(x, \underline{u}, \underline{u}', \bar{u}, \bar{u}') \tag{2.2}$$

With:

$$u(a) = A,\ u(b) = B$$

$$\overline{(u''(x))} = \bar{f}(x, u, u') = G(x, \underline{u}, \underline{u}', \bar{u}, \bar{u}') \tag{2.3}$$

With:

$$\bar{u}(a) = \bar{A}, \bar{u}(b) = \bar{B}$$

Where:

$$H(x, \underline{u}, \underline{u}', \bar{u}, \bar{u}) = Min\{f(x, z) : z \in [\underline{u}, \underline{u}', \bar{u}, \bar{u}']\} \tag{2.4}$$

$$G(x, \underline{u}, \underline{u}', \bar{u}, \bar{u}') = Max\{f(x, z) : z \in [\underline{u}, \underline{u}', \bar{u}, \bar{u}']\} \tag{2.5}$$

The system (2.4–2.5) can be written in parametric form as:

$$(u''(x, r)) = H(x, \underline{u}(x, r), \underline{u}'(x, r), \bar{u}(x, r), \bar{u}'(x, r)) \tag{2.6}$$

With:

$$\underline{u}(a,r) = \underline{A(r)}, \underline{u}(b,r) = \underline{B(r)}$$

$$\overline{u''(x,r)} = G(x, \underline{u}(x,r); \underline{u}'(x,r), \overline{u}(x,r), \overline{u}'(x,r)) \quad (2.7)$$

With:

$$\overline{u}(a,r) = \overline{A(r)}, \overline{u}(b,r) = \overline{B(r)}$$

To clarify what we explained above, we will take the second order FDE:

$$u''(x) = 6u'(x) - 9u(x) + [1+r, 3-r]x^2 \quad (2.8)$$

With:

$$u(0) = [2+r, 4-r], u(1) = [5+r, 7-r] \text{ and } r \in [0,1]$$

Equation (2.8) can be converted into a system of non-fuzzy differential equations as follows:

$$[u''(x)]_r = [6u'(x) - 9u(x)]_r + [1+r, 3-r]x^2 \quad (2.9)$$

With:

$$[u(0)]_r = [2+r, 4-r],\ [u(1)]_r = [5+r, 7-r]$$

Then, we get:

$$[u''(x)]_r = 6[u'(x)]_r - 9[u(x)]_r + [1+r, 3-r]x^2 \quad (2.10)$$

With:

$$[u(0)]_r = [2+r, 4-r],\ [u(1)]_r = [5+r, 7-r]$$

Then, we have:

$$[[u''(x)]_r^L, [u''(x)]_r^U] = [6[u'(x)]_r^L - 9[u(x)]_r^L + (1+r)x^2\ ,\ 6[u'(x)]_r^L - 9[u(x)]_r^L + (3-r)x^2] \quad (2.11)$$

With:

$$\left[[u(0)]_r^L, [u(0)]_r^U\right] = [2+r, 4-r]$$
$$\left[[u(1)]_r^L,\ [u(1)]_r^U\right] = [5+r, 7-r]$$

Then, we get the following system:

$$[u''(x)]_r^L = 6[u'(x)]_r^L - 9[u(x)]_r^L + (1+r)x^2 \quad (2.12)$$

With:

$$\begin{aligned} &[u(0)]_r^L = 2+r, [u(1)]_r^L = 5+r \\ &[u''(x)]_r^U = 6[u'(x)]_r^U - 9[u(x)]_r^U + (3-r)x^2 \end{aligned} \quad (2.13)$$

With:

$$[u(0)]_r^U = 4-r, [u(1)]_r^U = 7-r$$

This gives the crisp solutions:

$$[u(x)]_r^L = \left(\frac{52+25r}{27}\right)e^{3x} + \left(\frac{(126e^{-3}-52)+(18e^{-3}-25)r}{27}\right)xe^{3x} + \left(\frac{r+1}{9}\right)x^2 + \left(\frac{4r+4}{27}\right)x + \left(\frac{2r+2}{27}\right) \tag{2.14}$$

$$[u(x)]_r^U = \left(\frac{102-25r}{27}\right)e^{3x} + \left(\frac{(162e^{-3}-102)+(-18e^{-3}+25r)}{27}\right)xe^{3x} + \left(\frac{-r+3}{9}\right)x^2 + \left(\frac{-4r+12}{27}\right)x + \left(\frac{-2r+6}{27}\right) \tag{2.15}$$

Then, the fuzzy solution of equation (2.8) is:

$$[u(x)]_r = [[u(x)]_r^L, [u(x)]_r^U]$$

$$[u(x)]_r = \left[\left(\frac{52+25r}{27}\right)e^{3x} + \left(\frac{(126e^{-3}-52)+(18e^{-3}-25)r}{27}\right)xe^{3x} + \left(\frac{r+1}{9}\right)x^2 + \left(\frac{4r+4}{27}\right)x + \left(\frac{2r+2}{27}\right), \left(\frac{102-25r}{27}\right)e^{3x} + \left(\frac{(162e^{-3}-102)+(-18e^{-3}+25)r}{27}\right)e^{3x} + \left(\frac{-r+3}{9}\right)x^2 + \left(\frac{-4r+12}{27}\right)x + \left(\frac{-2r+6}{27}\right)\right] \tag{2.16}$$

3. Double Decomposition Method

To understand the double decomposition method, we consider the nonlinear two-point crisp boundary value problem [17]:

$$Lu(x) + Ru(x) + Nu(x) = g(x),\ x \in [\alpha_1, \alpha_2] \tag{3.1}$$

With:

$$u(\alpha_1) = \beta_1,\ u(\alpha_2) = \beta_2 \tag{3.2}$$

Where:

$L = \dfrac{d^2}{dx^2}$ is a second order linear differential operator which is invertible,
R is a linear operator follows the same assumptions of L but with order less than L ,
N is the non-linear operator,
g(x) is a given continuous function,
α_1, α_2, β_1, β_2 are real numbers.

By applying L^{-1} to the both sides of Equation (3.1), we will obtain:

$$u(x) = \theta(x) + L^{-1}g(x) - L^{-1}Ru(x) - L^{-1}Nu(x) \tag{3.3}$$

Where:

$$L^{-1}(*) = \iint (*)dxdx \tag{3.4}$$

$$\theta(x) = a + bx \tag{3.5}$$

Where a and b are real constants.

The Adomian approach is based on decomposing $u(x)$ of any equation and $Nu(x)$ into a sum of an infinite number of components defined by the decomposition series:

$$u = \sum_{n=0}^{\infty} u_n \tag{3.6}$$

$$Nu = \sum_{n=0}^{\infty} A_n \tag{3.7}$$

Where:

The components $u_n(x)$, $n \geq 0$ are to be determined in a recursive manner. The Adomian approach concerns itself by finding the components $u_0(x)$, $u_1(x)$, $u_2(x)$, … individually.

The components A_n, $n \geq 0$ depending on $u_0(x)$, $u_1(x)$, $u_2(x)$, … $u_n(x)$ are called the Adomian polynomials, and are obtained for the nonlinearity $Nu = f(u(x))$ as following [3, 16]:

$$A_0 = f(u_0) \tag{3.8}$$

$$A_1 = u_1 f'(u_0) \tag{3.9}$$

$$A_2 = u_2 f'(u_0) + \frac{u_1^2}{2!} f''(u_0) \tag{3.10}$$

$$A_3 = u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{u_1^3}{3!} f^{(3)}(u_0) \tag{3.11}$$

$$A_4 = u_4 f'(u0) + \left(u_1 u_3 + \frac{u_2^2}{2!} \right) f''(u_0) + \frac{u_{1u_2}^2}{2!} f^{(3)}(u_0) + \frac{u_1^4}{4!} f^{(4)}(u_0) \tag{3.12}$$

$$A_5 = u_5 f'(u_0) + (u_2 u_3 + u_1 u_4) f''(u_0) + \left(\frac{u_1 u_2^2}{2!} + \frac{u_3 u_1^2}{2!} \right) f^{(3)}(u_0) + \frac{u_2 u_1^3}{3!} f^{(4)}(u_0) + \frac{u_1^5}{5!} f^{(5)}(u_0) \tag{3.13}$$

$$\vdots$$

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \mu^n} \left[f\left(\sum_{k=0}^{\infty} \mu^k u_k \right) \right]_{\mu=0}, n = 0,1,2,\ldots \tag{3.14}$$

where μ is a grouping parameter of convenience.

Now, we decompose the term $\theta(x)$ in equation (3.3) into a sum of an infinite series as follows:

$$\theta = \sum_{n=0}^{\infty} \theta_n \tag{3.15}$$

Therefore, the equation (3.3) can be rewritten as follows:

$$\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \theta_n + L^{-1} g - L^{-1} R \sum_{n=0}^{\infty} u_n - L^{-1} \sum_{n=0}^{\infty} A_n \tag{3.16}$$

Where:

$$\theta_0 = a_0 + b_0 x \tag{3.17}$$

$$\theta_1 = a_1 + b_1 x \tag{3.18}$$

$$\theta_2 = a_2 + b_2 x \tag{3.19}$$

$$\theta_3 = a_3 + b_3 x \tag{3.20}$$

$$\vdots$$

$$\theta_n = a_n + b_n x \tag{3.21}$$

where a_n and b_n, $n \geq 0$ are real constants.

It is necessary to note that equation (3.21) is a special case of equation (3.5).

The solution steps of the double decomposition method can be derived from equation (3.16) as follows:

$$u_0 = \theta_0 + L^{-1}g(x) = a_0 + b_0 x + L^{-1}g(x) \tag{3.22}$$

$$u_1 = \theta_1 - L^{-1}R(u_0) - L^{-1}A_0 = a_1 + b_1 x + L^{-1}R(u_0) - L^{-1}A_0 \tag{3.23}$$

$$u_2 = \theta_2 - L^{-1}R(u_1) - L^{-1}A_1 = a_2 + b_2 x + L^{-1}R(u_1) - L^{-1}A_1 \tag{3.24}$$

$$u_3 = \theta_3 - L^{-1}R(u_2) - L^{-1}A_2 = a_3 + b_3 x + L^{-1}R(u_2) - L^{-1}A_2 \tag{3.25}$$

$$\vdots$$

$$u_n = \theta_n - L^{-1}R(u_{n-1}) - L^{-1}A_{n-1} = a_n + b_n x - L^{-1}R(u_{n-1}) - L^{-1}A_{n-1},\ n \geq 1 \tag{3.26}$$

It is necessary to note that the real constants a_n and b_n, $n \geq 0$ will be computed for every case of n by using the boundary conditions (equation 3.2).

Then, we have the approximate solution as follows:

$$\gamma_1(x) = u_0(x) \tag{3.27}$$

$$\gamma_2(x) = \gamma_1(x) + u_1(x) = u_0(x) + u_1(x) \tag{3.28}$$

$$\gamma_3(x) = \gamma_2(x) + u_2(x) = u_0(x) + u_1(x) + u_2(x) \tag{3.29}$$

$$\gamma_4(x) = \gamma_3(x) + u_3(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) \tag{3.30}$$

$$\vdots$$

$$\gamma_{n+1}(x) = \gamma_n(x) + u_n(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots + u_{n-1}(x) + u_n(x),\ n \geq 0 \tag{3.31}$$

This means, if we consider the first terms (say m) from the solution series (equation 3.6), then the approximate solution of problem (3.1) is:

$$u(x) \approx \gamma_m(x) = u_0(x) + u_1(x) + u_2(x) + \cdots + u_{m-1}(x) \tag{3.32}$$

It is important to note that with regard to the two-point fuzzy boundary value problem that we explained in the second section, we first convert this equation into two non-fuzzy equations (as we explained in the second section) and then apply the double decomposition method to each equation separately, to finally get the fuzzy solution.

4. Applied Examples

To explain the procedure of DDM, we will introduce some applied examples. We test the accuracy by computing the absolute errors:

$$[error]_r^L = \left| [u_{extract}(x)]_r^L - [u_{series}(x)]_r^L \right|$$

$$[error]_r^U = \left| [u_{extract}(x)]_r^U - [u_{series}(x)]_r^U \right|$$

Example 1: Consider the linear two-point fuzzy boundary value problem:

$$u''(x) = x - u(x),\ x \in [0,1]$$

With:

$$[u(0)]_r = [3 + r,\ 5 - r],\ [u(1)]_r = [r,\ 2 - r];\ r \in [0,1]$$

The fuzzy exact solution is:

$$[u(x)]_r = \left[[u(x)]_r^L, [u(x)]_r^U \right]$$

Where:

$$[u(x)]_r^L = x + (3+r)\cos x + (r-1)\csc 1 \sin x - (3+r)\cot 1 \sin x$$
$$[u(x)]_r^U = x + (5-r)\cos x + (1-r)\csc 1 \sin x - (5-r)\cot 1 \sin x$$

We will find the fuzzy series solution if , of course we can find the solution for every $r \in [0,1]$.

Lower bound of the fuzzy solution:

$$u''(x) = x - u(x);\ u(0) = 3.5,\ u(1) = 0.5$$

$$u(x) = \theta_n(x) + L^{-1}(x) - L^{-1}(u(x));\ \theta_n(x) = a_n + b_n x$$

$$u_0(x) = \theta_0(x) + L^{-1}(x)$$

$$u_n(x) = \theta_n(x) - L^{-1}(u_{n-1}(x)),\ n \geq 1$$

$$\gamma_1(x) = u_0(x);\ \ u_0(x) = a_0 + b_0 x + \frac{x^3}{6}$$

By using the boundary conditions, we get: $a_0 = \frac{7}{2},\ b_0 = -\frac{19}{6}$

Therefore, we have:

$$u_0(x) = \frac{7}{2} - \frac{19}{6}x + \frac{x^3}{6}$$

$$\gamma_1(x) = \frac{7}{2} - \frac{19}{6}x + \frac{x^3}{6}$$

$$\gamma_2(x) = \gamma_1(x) + u_1(x),\ u_1(x) = \theta_1(x) - L^{-1}(u_0(x))$$

$$u_1(x) = a_1 + b_1 x - L^{-1}\left(\frac{7}{2} - \frac{19}{6}x + \frac{x^3}{6}\right)$$

$$u_1(x) = a_1 + b_1 x - \frac{7}{4}x^2 + \frac{19}{36}x^3 - \frac{1}{120}x^5$$

$$\gamma_2(x) = a_1 + b_1 x + \frac{7}{2} - \frac{19}{6}x - \frac{7}{4}x^2 + \frac{25}{36}x^3 - \frac{1}{120}x^5$$

By using the boundary conditions, we get: $a_1 = 0,\ b_1 = \frac{443}{360}$

Therefore, we have:

$$u_1(x) = \frac{443}{360}x - \frac{7}{4}x^2 + \frac{19}{36}x^3 - \frac{1}{120}x^5$$

$$\gamma_2(x) = \frac{7}{2} - \frac{697}{360}x - \frac{7}{4}x^2 + \frac{25}{36}x^3 - \frac{1}{120}x^5$$

$$\gamma_3(x) = \gamma_2(x) + u_2(x);\ u_2(x) = \theta_2(x) - L^{-1}(\mu_1(x))$$

$$u_2(x) = a_2 + b_2x - L^{-1}\left(\frac{443}{360}x - \frac{7}{4}x^2 + \frac{19}{36}x^3 - \frac{1}{120}x^5\right)$$

$$u_2(x) = a_2 + b_2x - \frac{443}{2160}x^3 + \frac{7}{48}x^4 - \frac{19}{720}x^5 + \frac{1}{5040}x^7$$

$$\gamma_3(x) = a_2 + b_2x + \frac{7}{2} - \frac{697}{360}x - \frac{7}{4}x^2 + \frac{1057}{2160}x^3 + \frac{7}{48}x^4 - \frac{5}{144}x^5 + \frac{1}{5040}x^7$$

By using the boundary conditions, we get: $a_2 = 0,\ b_2 = \frac{323}{3780}$

Therefore, we have:

$$u_2(x) = \frac{323}{3780}x - \frac{443}{2160}x^3 + \frac{7}{48}x^4 - \frac{19}{720}x^5 + \frac{1}{5040}x^7$$

$$\gamma_3(x) = \frac{7}{2} - \frac{13991}{7560}x - \frac{7}{4}x^2 + \frac{1057}{2160}x^3 + \frac{7}{48}x^4 - \frac{5}{144}x^5 + \frac{1}{5040}x^7$$

$$\gamma_4(x) = \gamma_3(x) + u_3(x); u_3(x) = \theta_3(x) - L^{-1}(u_2(x))$$

$$u_3(x) = a_3 + b_3x - L^{-1}\left(\frac{323}{3780}x - \frac{443}{2160}x^3 + \frac{7}{48}x^4 - \frac{19}{720}x^5 + \frac{1}{5040}x^7\right)$$

$$u_3(x) = a_3 + b_3x - \frac{323}{22680}x^3 + \frac{443}{43200}x^5 - \frac{7}{1440}x^6 + \frac{19}{30240}x^7 - \frac{1}{362880}x^9$$

$$\gamma_4(x) = a_3 + b_3x + \frac{7}{2} - \frac{13991}{7560}x - \frac{7}{4}x^2 + \frac{21551}{45360}x^3 + \frac{7}{48}x^4 - \frac{1057}{43200}x^5 - \frac{7}{1440}x^6 + \frac{5}{6048}x^7 - \frac{1}{362880}x^9$$

By using the boundary conditions, we get: $a_3 = 0,\ b_3 = \frac{4973}{604800}$

Therefore, we have:

$$u_3(x) = \frac{4973}{604800}x - \frac{323}{22680}x^3 + \frac{443}{43200}x^5 - \frac{7}{1440}x^6 + \frac{19}{30240}x^7 - \frac{1}{362880}x^9$$

$$\gamma_4(x) = \frac{7}{2} - \frac{1114307}{604800}x - \frac{7}{4}x^2 + \frac{21551}{45360}x^3 + \frac{7}{48}x^4 - \frac{1057}{43200}x^5 - \frac{7}{1440}x^6 + \frac{5}{6048}x^7 - \frac{1}{362880}x^9$$

$$\gamma_5(x) = \gamma_4(x) + u_4(x); u_4(x) = \theta_4(x) - L^{-1}(u_3(x))$$

$$u_4(x) = a_4 + b_4 x - L^{-1}\left(\frac{4973}{604800}x - \frac{323}{22680}x^3 + \frac{443}{43200}x^5 - \frac{7}{1440}x^6 + \frac{19}{30240}x^7 - \frac{1}{362880}x^9\right)$$

$$u_4(x) = a_4 + b_4 x - \frac{4973}{3628800}x^3 + \frac{323}{453600}x^5 - \frac{443}{1814400}x^7 + \frac{1}{11520}x^8 - \frac{19}{2177280}x^9 + \frac{1}{39916800}x^{11}$$

$$\gamma_5(x) = a_4 + b_4 x + \frac{7}{2} - \frac{1114307}{604800}x - \frac{7}{4}x^2 + \frac{1719107}{3628800}x^3 + \frac{7}{48}x^4 - \frac{21551}{907200}x^5 - \frac{7}{1440}x^6 + \frac{151}{259200}x^7 + \frac{1}{11520}x^8 - \frac{5}{435453}x^9 + \frac{1}{39916800}x^{11}$$

By using the boundary conditions, we get: $a_4 = 0,\ b_4 = \frac{94021}{114048000}$

Therefore, we have:

$$u_4(x) = \frac{94021}{11404800}x - \frac{4973}{3628800}x^3 + \frac{323}{453600}x^5 - \frac{443}{1814400}x^7 + \frac{1}{11520}x^8 - \frac{19}{2177280}x^9 + \frac{1}{39916800}x^{11}$$

$$\gamma_5(x) = \frac{7}{2} - \frac{79392263}{43110144}x - \frac{7}{4}x^2 + \frac{1719107}{3628800}x^3 + \frac{7}{48}x^4 - \frac{21551}{907200}x^5 - \frac{7}{1440}x^6 + \frac{151}{259200}x^7 + \frac{1}{11520}x^8 - \frac{5}{435456}x^9 + \frac{1}{39916800}x^{11}$$

$$\vdots$$

Therefore, the lower bound of the fuzzy series solution is:

$$[u(x)]_r^L \approx \gamma_5(x)$$

$$[u(x)]_r^L = \frac{7}{2} - \frac{79392263}{43110144}x - \frac{7}{4}x^2 + \frac{1719107}{3628800}x^3 + \frac{7}{48}x^4 - \frac{21551}{907200}x^5 - \frac{7}{1440}x^6 + \frac{151}{259200}x^7 + \frac{1}{11520}x^8 - \frac{5}{435456}x^9 + \frac{1}{39916800}x^{11}$$

Upper bound of the fuzzy solution:

$$u''(x) = x - u(x);\ u(0) = 4.5\ ,\ u(1) = 1.5$$

In the same manner that we followed in the first part of this example, we can find:

$$\gamma_1(x) = \frac{9}{2} - \frac{19}{6}x + \frac{x^3}{6}$$

$$\gamma_2(x) = \frac{9}{2} - \frac{517}{360}x - \frac{9}{4}x^2 + \frac{25}{36}x^3 - \frac{1}{120}x^5$$

$$\gamma_3(x) = \frac{9}{2} - \frac{1237}{945}x - \frac{9}{4}x^2 + \frac{877}{2160}x^3 + \frac{9}{48}x^4 - \frac{25}{720}x^5 + \frac{1}{5040}x^7$$

$$\gamma_4(x) = \frac{9}{2} - \frac{784187}{604800}x - \frac{9}{4}x^2 + \frac{1091}{2835}x^3 + \frac{9}{48}x^4 - \frac{877}{43200}x^5 - \frac{9}{1440}x^6 + \frac{8}{6048}x^7 - \frac{1}{362880}x^9$$

$$\gamma_5(x) = \frac{9}{2} - \frac{2939327713}{184757760}x - \frac{9}{4}x^2 + \frac{1388987}{3628800}x^3 + \frac{9}{48}x^4 - \frac{1091}{56700}x^5 - \frac{9}{1440}x^6 + \frac{877}{1814400}x^7 + \frac{9}{80640}x^8 - \frac{5}{435456}x^9 + \frac{1}{39916800}x^{11}$$

$$\vdots$$

Therefore, the upper bound of the fuzzy series solution is:

$$[u(x)]_r^U \approx \gamma_5(x)$$

$$[u(x)]_r^U = \frac{9}{2} - \frac{239327713}{184757760}x - \frac{9}{4}x^2 + \frac{1388987}{3628800}x^3 + \frac{9}{48}x^4 - \frac{1091}{56700}x^5 - \frac{9}{1440}x^6 + \frac{877}{1814400}x^7 + \frac{9}{80640}x^8 - \frac{5}{435456}x^9 + \frac{1}{39916800}x^{11}$$

Then, we have the following fuzzy series solution:

$$[u(x)]_r = \left[[u(x)]_r^L, [u(x)]_r^U\right]$$

$$[u(x)]_r = \left[\frac{7}{2} - \frac{79392263}{43110144}x - \frac{7}{4}x^2 + \frac{1719107}{3628800}x^3 + \frac{7}{48}x^4 - \frac{21551}{907200}x^5 - \frac{7}{1440}x^6 + \frac{151}{259200}x^7 + \frac{1}{11520}x^8 - \frac{5}{435456}x^9 + \frac{1}{39916800}x^{11}, \frac{9}{2} - \frac{239327713}{184757760}x - \frac{9}{4}x^2 + \frac{1388987}{3628800}x^3 + \frac{9}{48}x^4 - \frac{1091}{56700}x^5 - \frac{9}{1440}x^6 + \frac{877}{1814400}x^7 + \frac{9}{80640}x^8 - \frac{5}{435456}x^9 + \frac{1}{39916800}x^{11}\right]$$

Numerical results for this problem can be found in the Table 1.

Example 2: Consider the nonlinear two-point fuzzy boundary value problem:

$$u''(x) = -(u'(x))^2, \; x \in [0,2]$$

With:

$$[u(0)]_r = [r, 2 - r], \; [u(2)]_r = [1 + r, 3 - r]; \; r \in [0,1].$$

Table 1. Numerical results for example 1.

x	$[u_{series}(x)]_r^L$	$[error]_r^L$	$[u_{series}(x)]_r^U$	$[error]_r^U$
0	3.500000000000000	0	4.500000000000000	0
0.1	3.298826638405800	9.12e-6	4.348365372544997	1.38e-5
0.2	3.065692461813599	1.73e-5	4.154283696960543	2.62e-5
0.3	2.803926760812489	2.39e-5	3.920694434856567	3.61e-5
0.4	2.517144752079745	2.81e-5	3.650931567389507	4.24e-5
0.5	2.209211400275634	2.95e-5	3.348690198407133	4.46e-5
0.6	1.884202765178947	2.80e-5	3.017989580334067	4.24e-5
0.7	1.546365263519024	2.38e-5	2.663132937253816	3.61e-5
0.8	1.200073253413404	1.73e-5	2.288664488096421	2.62e-5
0.9	0.849785363711897	9.11e-6	1.899324097232523	1.38e-5
1	0.500000000927856	9.28e-10	1.500000000154643	1.55e-10

The fuzzy exact solution is:

$$[u(x)]_r = \left[[u(x)]_r^L , [u(x)]_r^U \right]$$

Where:

$$[u(x)]_r^L = \ln\left(x + \frac{2}{e-1} \right) + r - \ln\left(\frac{2}{e-1} \right)$$
$$[u(x)]_r^U = \ln\left(x + \frac{2}{e-1} \right) + 2 - r - \ln\left(\frac{2}{e-1} \right)$$

We will find the fuzzy series solution if $r = 0.5$, of course we can find the solution for every $r \in [0,1]$.

First, we find the Adomian polynomials for the function $(u'(x))^2$ as follows:

$$A_0 = (u_0'(x))^2$$
$$A_1 = 2u_0'(x)u_1'(x)$$
$$A_2 = 2u_0'(x)u_2'(x) + (u_1'(x))^2$$
$$A_3 = 2u_0'(x)u_3'(x) + 2u_1'(x)u_2'(x)$$
$$A_4 = 2u_0'(x)u_4'(x) + (u_2'(x))^2 + 2u_1'(x)u_3'(x)$$
$$A_5 = 2u_0'(x)u_5'(x) + 2u_2'(x)u_3'(x) + 2u_1'(x)u_4'(x)$$
$$\vdots$$

Lower bound of the fuzzy solution:

$$u''(x) = -(u'(x))^2;\ u(0) = 0.5,\ u(2) = 1.5$$

$$u(x) = \theta_n(x) - \mathrm{L}^{-1}((u_0'(x))^2);\ \theta_n(x) = a_n + b_n x$$

$$u_0(x) = \theta_0(x)$$

$$u_n(x) = \theta_n(x) - L^{-1}(A_{n-1}),\ n \geq 1$$

$$\gamma_1(x) = u_0(x);\ u_0(x) = a_0 + b_0 x$$

By using the boundary conditions, we get: $a_0 = \frac{1}{2}$, $b_0 = \frac{1}{2}$

Therefore, we have:

$$u_0(x) = \frac{1}{2} + \frac{1}{2}x$$
$$\gamma_1(x) = \frac{1}{2} + \frac{1}{2}x$$
$$\gamma_2(x) = \gamma_1(x) + u_1(x);\ u_1(x) = \theta_1(x) - L^{-1}(A_0)$$
$$u_1(x) = a_1 + b_1 x - L^{-1}((u_0'(x))^2)$$
$$u_1(x) = a_1 + b_1 x - \frac{1}{8}x^2$$
$$\lambda_2(x) = a_1 + b_1 x + \frac{1}{2} + \frac{1}{2}x - \frac{1}{8}x^2$$

By using the boundary conditions, we get: $a_1 = 0,\ b_1 = \frac{1}{4}$

Therefore, we have:

$$u_1(x) = \frac{1}{4}x - \frac{1}{8}x^2$$

$$\gamma_2(x) = \frac{1}{2} + \frac{3}{4}x - \frac{1}{8}x^2$$

$$\gamma_3(x) = \gamma_2(x) + u_2(x); u_2(x) = \theta_2(x) - L^{-1}(A_1)$$

$$u_2(x) = a_2 + b_2 x - L^{-1}(2u_0'(x)u_1'(x))$$

$$u_2(x) = a_2 + b_2 x - \frac{1}{8}x^2 + \frac{1}{24}x^3$$

$$\gamma_3(x) = a_2 + b_2 x + \frac{1}{2} + \frac{3}{4}x - \frac{1}{4}x^2 + \frac{1}{24}x^3$$

By using the boundary conditions, we get: $a_2 = 0,\ b_2 = \frac{1}{12}$

Therefore, we have:

$$u_2(x) = \frac{1}{12}x - \frac{1}{8}x^2 + \frac{1}{24}x^3$$

$$\gamma_3(x) = \frac{1}{2} + \frac{5}{6}x - \frac{1}{4}x^2 + \frac{1}{24}x^3$$

$$\gamma_4(x) = \gamma_3(x) + u_3(x); u_3(x) = \theta_3(x) - L^{-1}(A_2)$$

$$u_3(x) = a_3 + b_3 x - L^{-1}(2u_0'(x)u_2'(x) + (u_1'(x))^2)$$

$$u_3(x) = a_3 + b_3 x - \frac{7}{96}x^2 + \frac{1}{16}x^3 - \frac{1}{64}x^4$$

$$\gamma_4(x) = a_3 + b_3 x + \frac{1}{2} + \frac{5}{6}x - \frac{31}{96}x^2 + \frac{5}{48}x^3 - \frac{1}{64}x^4$$

By using the boundary conditions, we get: $a_3 = 0,\ b_3 = \frac{1}{48}$

Therefore, we have:

$$u_3(x) = \frac{1}{48}x - \frac{7}{96}x^2 + \frac{1}{16}x^3 - \frac{1}{64}x^4$$

$$\gamma_4(x) = \frac{1}{2} + \frac{41}{48}x - \frac{31}{96}x^2 + \frac{5}{48}x^3 - \frac{1}{64}x^4$$

$$\gamma_5(x) = \gamma_4(x) + u_4(x); u_4(x) = \theta_4(x) - L^{-1}(A_3)$$

$$u_4(x) = a_4 + b_4 x - L^{-1}(2u_0'(x)u_3'(x) + 2u_1'(x)u_2'(x))$$

$$u_4(x) = a_4 + b_4 x - \frac{1}{32}x^2 + \frac{5}{96}x^3 - \frac{1}{32}x^4 + \frac{1}{160}x^5$$

$$\gamma_5(x) = a_4 + b_4 x + \frac{1}{2} + \frac{41}{48}x - \frac{17}{48}x^2 + \frac{5}{32}x^3 - \frac{3}{64}x^4 + \frac{1}{160}x^5$$

By using the boundary conditions, we get: $a_4 = 0,\ b_4 = \frac{1}{240}$

Therefore, we have:

$$u_4(x) = \frac{1}{240}x - \frac{1}{32}x^2 + \frac{5}{96}x^3 - \frac{1}{32}x^4 + \frac{1}{160}x^5$$
$$\gamma_5(x) = \frac{1}{2} + \frac{103}{120}x - \frac{17}{48}x^2 + \frac{5}{32}x^3 - \frac{3}{64}x^4 + \frac{1}{160}x^5$$
$$\gamma_6(x) = \gamma_5(x) + u_5(x); u_5(x) = \theta_5(x) - L^{-1}(A_4)$$
$$u_5(x) = a_5 + b_5 x - L^{-1}(2u_0'(x)u_4'(x) + u_2'(x))^2 + 2u_1'(x)u_3'(x))$$
$$u_5(x) = a_5 + b_5 x - \frac{31}{2880}x^2 + \frac{1}{32}x^3 - \frac{13}{384}x^4 + \frac{1}{64}x^5 - \frac{1}{384}x^6$$
$$\gamma_6(x) = a_5 + b_5 x + \frac{1}{2} + \frac{103}{120}x - \frac{1051}{2880}x^2 + \frac{3}{16}x^3 - \frac{31}{384}x^4 + \frac{7}{320}x^5 - \frac{1}{384}x^6$$

By using the boundary conditions, we get: $a_5 = 0,\ b_5 = \frac{1}{1440}$

Therefore, we have:

$$u_5(x) = \frac{1}{1440}x - \frac{31}{2880}x^2 + \frac{1}{32}x^3 - \frac{13}{384}x^4 + \frac{1}{64}x^5 - \frac{1}{384}x^6$$
$$\gamma_6(x) = \frac{1}{2} + \frac{1237}{1440}x - \frac{1051}{2880}x^2 + \frac{3}{16}x^3 - \frac{31}{384}x^4 + \frac{7}{320}x^5 - \frac{1}{384}x^6$$
$$\gamma_7(x) = \gamma_6(x) + u_6(x); u_6(x) = \theta_6(x) - L^{-1}(A_5)$$
$$u_6(x) = a_6 + b_6 x - L^{-1}(2u_0'(x)u_5'(x) + 2u_2'(x)u_3'(x) + 2u_1'(x)u_4'(x))$$
$$u_6(x) = a_6 + b_6 x - \frac{1}{320}x^2 + \frac{43}{2880}x^3 - \frac{5}{192}x^4 + \frac{1}{48}x^5 - \frac{1}{128}x^6 + \frac{1}{186}x^7$$
$$\gamma_7(x) = a_6 + b_6 x + \frac{1}{2} + \frac{1237}{1440}x - \frac{53}{144}x^2 + \frac{583}{2880}x^3 - \frac{41}{384}x^4 + \frac{41}{960}x^5 - \frac{1}{96}x^6 + \frac{1}{896}x^7$$

By using the boundary conditions, we get: $a_6 = 0,\ b_6 = \frac{1}{10080}$

Therefore, we have:

$$u_6(x) = \frac{1}{10080}x - \frac{1}{320}x^2 + \frac{43}{2880}x^3 - \frac{5}{192}x^4 + \frac{1}{48}x^5 - \frac{1}{128}x^6 + \frac{1}{896}x^7$$
$$\gamma_7(x) = \frac{1}{2} + \frac{433}{504}x - \frac{53}{144}x^2 + \frac{583}{2880}x^3 - \frac{41}{384}x^4 + \frac{41}{960}x^5 - \frac{1}{96}x^6 + \frac{1}{896}x^7$$
$$\vdots$$

Therefore, the lower bound of the fuzzy series solution is:

$$[u(x)]_r^L \approx \gamma_7(x)$$
$$[u(x)]_r^L = \frac{1}{2} + \frac{433}{504}x - \frac{53}{144}x^2 + \frac{83}{2880}x^3 - \frac{41}{384}x^4 + \frac{41}{960}x^5 - \frac{1}{96}x^6 + \frac{1}{896}x^7$$

Upper bound of the fuzzy solution:

$$u''(x) = -(u'(x))^2;\ u(0) = 1.5,\ u(2) = 2.5$$

In the same manner that we followed in the first part of this example, we can find:

$$\gamma_1(x)=\frac{3}{2}+\frac{1}{2}x$$
$$\gamma_2(x)=\frac{3}{2}+\frac{3}{4}x-\frac{1}{8}x^2$$
$$\gamma_3(x)=\frac{3}{2}+\frac{5}{6}x-\frac{1}{4}x^2+\frac{1}{24}x^3$$
$$\gamma_4(x)=\frac{3}{2}+\frac{41}{48}x-\frac{31}{96}x^2+\frac{5}{48}x^3-\frac{1}{64}x^4$$
$$\gamma_5(x)=\frac{3}{2}+\frac{103}{120}x-\frac{17}{48}x^2+\frac{5}{32}x^3-\frac{3}{64}x^4+\frac{1}{160}x^5$$
$$\gamma_6(x)=\frac{3}{2}+\frac{1237}{1440}x-\frac{1051}{2880}x^2+\frac{3}{16}x^3-\frac{31}{384}x^4+\frac{7}{320}x^5-\frac{1}{384}x^6$$
$$\gamma_7(x)=\frac{3}{2}+\frac{433}{504}x-\frac{53}{144}x^2+\frac{583}{2880}x^3-\frac{41}{384}x^4+\frac{41}{960}x^5-\frac{1}{96}x^6+\frac{1}{896}x^7$$
$$\vdots$$

Therefore, the upper bound of the fuzzy series solution is:

$$[u(x)]_r^U \approx \gamma_7(x)$$
$$[u(x)]_r^U=\frac{3}{2}+\frac{433}{504}x-\frac{53}{144}x^2+\frac{583}{2880}x^3-\frac{41}{384}x^4+\frac{41}{960}x^5-\frac{1}{96}x^6+\frac{1}{896}x^7$$

Then, we have the following fuzzy series solution:

$$[u(x)]_r=\left[[u(x)]_r^L,[u(x)]_r^U\right]$$
$$[u(x)]_r=\left[\frac{1}{2}+\frac{433}{504}x-\frac{53}{144}x^2+\frac{583}{2880}x^2+\frac{41}{384}x^4+\frac{41}{960}x^5-\frac{1}{96}x^6+\frac{1}{896}x^7,\frac{3}{2}+\frac{433}{504}x\right.$$
$$\left.-\frac{53}{144}x^2+\frac{583}{2880}x^3-\frac{41}{384}x^4+\frac{41}{960}x^5-\frac{1}{96}x^6+\frac{1}{896}x^7\right]$$

Numerical results for this problem can be found in the following table:

Table 2. Numerical results for example 2.

x	$[u_{series}(x)]_r^L$	$[error]_r^L$	$[u_{series}(x)]_r^U$	$[error]_r^U$
0	0.500000000000000	0	1.500000000000000	0
0.2	0.658564800000000	2.79e-7	1.658564800000000	2.79e-7
0.4	0.795380622222222	1.39e-5	1.795380622222222	1.39e-5
0.6	0.915729933333333	5.29e-6	1.915729933333333	5.29e-6
0.8	1.023155200000000	1.80e-5	2.023155200000000	1.80e-5
1	1.120138888888889	2.44e-5	2.120138888888889	2.44e-5
1.2	1.208519466666666	6.40e-6	2.208519466666666	6.40e-6
1.4	1.289715400000000	1.26e-5	2.289715400000000	1.26e-5
1.6	1.364829155555555	1.06e-5	2.364829155555555	1.06e-5
1.8	1.434703200000000	1.54e-6	2.434703200000000	1.54e-6
2	1.500000000000000	0	2.500000000000000	0

Example 3: Consider the fuzzy Painleve Equation I:

$$u''(x) = x + 6(u(x))^2,\ x \in [0,1]$$

With:

$$[u(0)]_r = [0.5r + 1, -0.5r + 2],\ [u(1)]_r = [0.5r + 3, -0.5r + 4],\ r \in [0,1]$$

First, we apply the double decomposition method to get:

$$u(x) = \theta_n(x) + L^{-1}(x) + L^{-1}(6(u(x))^2);\ \theta_n(x) = a_n + b_n x$$

$$u_0(x) = \theta_0(x) + L^{-1}(x) = \theta_0(x) + \frac{1}{6}x^3$$

$$u_n(x) = \theta_n(x) + L^{-1}(A_{n-1}), n \geq 1$$

Now, we find the Adomian polynomials for the function $6(u(x))^2$ as follows:

$$A_0 = 6(u_0(x))^2$$
$$A_1 = 12u_0(x)u_1(x)$$
$$A_2 = 12u_0(x)u_2(x) + 6(u_1(x))^2$$
$$A_3 = 12u_0(x)u_3(x) + 12u_1(x)u_2(x)$$
$$A_4 = 12u_0(x)u_4(x) + 6(u_2(x))^2 + 12u_1(x)u_3(x)$$
$$A_5 = 12u_0(x)u_5(x) + 12u_5(x)u_3(x) + 12u_1(x)u_4(x)$$
$$\vdots$$

In the same manner that we followed in the previous examples, we can find the fuzzy approximate solution for this problem if $r = 0.5$, as follows:

$$[u(x)]_r = \left[[u(x)]_r^L, [u(x)]_r^U \right]$$

Where:

$$[u(x)]_r^L = \frac{5}{4} + \frac{68989740030}{3487787229}x + \frac{75}{16}x^2 - \frac{1465}{63}x^3 - \frac{786017}{60480}x^4 + \frac{279}{32}x^5 + \frac{270409}{75600}x^6 + \frac{434}{378}x^7 + \frac{167}{672}x^8 + \frac{121}{1440}x^9 + \frac{33}{10080}x^{10} + \frac{781}{277200}x^{11} + \frac{1}{26208}x^{13}$$

$$[u(x)]_r^U = \frac{7}{4} + \frac{8317770101}{215694943}x + \frac{147}{16}x^2 - \frac{19787}{360}x^3 - \frac{875771}{60480}x^4 + \frac{2723}{160}x^5 + \frac{365287}{75600}x^6 + \frac{2656}{1890}x^7 + \frac{233}{672}x^8 + \frac{121}{1440}x^9 + \frac{33}{7200}x^{10} + \frac{781}{277200}x^{11} + \frac{1}{26208}x^{13}$$

5. Conclusion

In this work, we have used double decomposition method to find approximate-analytical solutions for the two-point fuzzy boundary value problems. The accurate results obtained in this research will encourage the researchers to improve this method to include other types of fuzzy differential equations, such as multi-point fuzzy boundary value problems and fuzzy partial differential equations.

References

[1] T. Allahviranloo, L. Jamshidi, Solution of fuzzy differential equations under generalized differentiability by Adomian decomposition method, *Iranian Journal of Optimization,* 1, 57–75, 2009.

[2] L. Wang, S. Guo, Adomian method for second order fuzzy differential equation, World Academy of Science, *Engineering and Technology*, 52, 979–982, 2011.

[3] J. S. Duan, R. Rach, D. Baleanu, A. M. Wazwaz, A review of the Adomian decomposition method and its applications to fractional differential equations, *Commun. Frac. Calc.*, 3(2), 73–99, 2012.

[4] S. Narayanamoorthy, T. L. Yookesh, An Adomian decomposition method to solve linear fuzzy differential equations, Proceeding of the International Conference on Mathematical Methods and Computation, Jamal Mohamed College, Tiruchirapalli, India, 13–14 February 2014.

[5] A. F. Jameel, Numerical and approximate-analytical solutions of fuzzy initial value problems, Ph.D. Thesis, School of Quantitative Sciences, University Utara Malaysia, Malaysia, 2015.

[6] M. Paripour, E. Hajilou, A. Hajilou, H. Heidari, Application of Adomian decomposition method to solve hybrid fuzzy differential equations, *Journal of Taibah University for Science*, 9, 95–103, 2015. https://doi.org/10.1016/j.jtusci.2014.06.002

[7] M. H. Suhhiem, Fuzzy artificial neural network for solving fuzzy and non-fuzzy differential equations, Ph.D. Thesis, College of Sciences, University of Al-Mustansiriyah, Iraq, 2016.

[8] S. Biswas, S. Banerjee, T. K. Roy, Solving intuitionistic fuzzy differential equations with linear differential operator by Adomian decomposition method, 3rd Int. IFS Conf., 29 Aug-1 Sep 2016, Mersin, Turkey. Notes on Intuitionistic Fuzzy Sets, 22(4), 25–41, 2016.

[9] S. Biswas, T. K. Roy, Adomian decomposition method for fuzzy differential equations with linear differential operator, *Journal of Information and Computing Science*, 11(4), 243–250, 2016.

[10] A. K. Ateeah, Approximate solution for fuzzy differential algebraic equations of fractional order using Adomian decomposition method, *Ibn Al-Haitham Journal for Pure and Applied Sciences*, 30(2), 202–213, 2017.

[11] H. G. Citil, On the exact solutions and the approximate solutions by Adomian decomposition method of the second-order linear fuzzy initial value problems using the generalized Differentiability, *Scholars Journal of Physics, Mathematics and Statistics*, 4(4), 145–161, 2017. https://doi.org/10.21276/sjpms.2017.4.4.1

[12] H. G. Citil, On the exact and the approximate solutions of second-order fuzzy initial value problems with constant coefficients, *Malaya Journal of Matematik*, 6(1), 61–68, 2018. https://doi.org/10.26637/MJM0601/0009

[13] H. G. Citil, Comparisons of the exact and the approximate solutions of second-order fuzzy linear boundary value problems, *Miskolc Mathematical Notes*, 20(2), 823–837, 2019. https://doi.org/10.18514/MMN.2019.2627

[14] S. Askari, T. Allahviranloo, S. Abbasbandy, Solving fuzzy fractional differential equations by Adomian decomposition method used in optimal control theory, *International Transaction Journal of Engineering, Management, & Applied Sciences & Technologies*, 10(12), 1–11, 2019. https://doi.org/10.6084/m9.figshare.11110514

[15] H. A. Sabr, B. N. Abood, M. H. Suhhiem, Fuzzy homotopy anaysis method for solving fuzzy autonomous differential equation, *Ratio Mathematica*, 40, 191–212, 2021. https://doi.org/10.23755/rm.v40i1.589

[16] M. H. Suhhiem, R. I. Khwayyit, Semi analytical solution for fuzzy autonomous differential equations, *International Journal of Analysis and Applications*, 20: 61, 1–18, 2022. https://doi.org/10.28924/2291-8639-20-2022-61

[17] N. AL-Zaid, A. AL-Refaidi, H. Bakodah, M. AL-Mazmumy, Solution of second- and higher-order nonlinear two-point boundary-value problems using double decomposition method, *Mathematics* 2022, 10, 3519, 1–15, 2022. https://doi.org/10.3390/math10193519