Results in Nonlinear Analysis 7 (2024) No. 4, 115–122
https://doi.org/10.31838/rna/2024.07.04.012
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

# A new structure of spaces using the framework of generalized Δ-operator method

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The focus of the study in the this paper is to introduce the space $\mathcal{L}_s^{\vartheta}\left(p, \Delta_g^w\right)$. The corresponding completeness property will be determined. Also, various topological properties will be enlightened.

*Key words and phrases:* Beppo–Levi's space; Paranormed spaces; Lacunary sequences

*Mathematics Subject Classification (2020):* 46B45; 46A45; 46B99.

## 1. Introduction and Background

A sequence space is a function space whose entries are functions from non-negative integers numbers $\mathbb{N}$ to the field $K$ of real numbers $\mathbb{R}$ or of complex numbers $\mathbb{C}$. An interesting topological structures of Banach spaces is visible through Opial property, Fatou property along with their generalization. It is the Opial structure which plays a vital role in Banach spaces as such spaces with this property attain the weak fixed point property. Recently, the author in [15], $\ell_p$ $(1 < p < \infty)$ attains this structure but the set $L_p[0,2\pi]$, $(p \neq 2, 1 < p < \infty)$ does not. Also, the author in [8] proved that any infinite dimensional Banach space has an equivalent norm satisfying the Opial property. Further, in [19], the author has considered the uniform Opial property for Banach spaces in detail. Later, it was studied by various authors as can be seen in [1,2, 13, 17], and many others.

Let $\vartheta = (\kappa_r)$ be a sequence of natural numbers with $\kappa_0 = 0$, $0 < \kappa_r < \kappa_{r+1}$ and $h_r = \kappa_r - \kappa_{r-1} \to \infty$ as $r \to \infty$. Then $\vartheta$ is called a lacunary sequence. The intervals computed by $\vartheta$ are abbreviated by $\mathfrak{J}_r = (\kappa_{r-1}, \kappa_r]$ and the quotient $\dfrac{\kappa_r}{\kappa_{r-1}}$ will be symbolized by $q_r$.

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*Received June 23, 2024; Accepted September 18, 2024; Online November 20, 2024*

Later, it was studied by different authors as can be seen in [4], [9] and many others for different domains. In [10], [11], the author examined geometric structures connecting lacunary sequences with Cesàro space by equipping Luxemburg norm.

In [12], the spaces $\mathfrak{T}(\Delta)$ were studied and is defined as follows:

$$\mathfrak{T}(\Delta) = \{v = (v_i) \in \Lambda : (\Delta v_i) \in \mathfrak{T}\}$$

where $\mathfrak{T} \in \{\ell_\infty, c, \mathfrak{C}_0\}$ and $\Delta v_i = v_i - v_{i-1}$.

Next for integer $w \geq 0$, the author in [5] had studied the following space:

$$\Delta^w(\mathfrak{T}) = \left\{ v = (v_k) : \left(\Delta^w v\right) \in \mathfrak{T} \right\}, \text{ for } \mathfrak{T} = \ell_\infty, c \text{ and } \mathfrak{C}_0,$$

where $\Delta^w v_i = \Delta^{w-1} v_i - \Delta^{w-1} v_{i+1}$ for all $i \in \mathbb{N}$. Also, let $g = (g_j)$ be any fixed sequence of non-zero complex numbers, then as in [6], we have

$$\Delta_g^w(\mathfrak{T}) = \left\{ v = (v_j) \in \Lambda : (\Delta_g^w v_j) \in \mathfrak{T} \right\},$$

where

$$\Delta_g^w v_j = \Delta_g^{w-1} v_j - \Delta_g^{w-1} v_{j+1} = \sum_{\mu=0}^{w} (-1)^\mu \binom{w}{\mu} g_{j+\mu} v_{j+\mu} \ \forall j \in \mathbb{N}.$$

It is shown that the space $\Delta_g^s(\mathfrak{T})$ is Banach under the norm

$$\|v\|_\Delta = \sum_{i=1}^{w} |g_i v_i| + \left\|{}_g^w v\right\|_\infty.$$

Many interesting structures towards this space can be searched in [4, 7, 14, 17, 23] and many others.

As in [20], by ε-separated sequence with $\varepsilon > 0$, we mean a sequence $\{\zeta_i\} \subset \Lambda$ such that

$$\operatorname{sep}\{\zeta_j\} = \inf\left\{ \|\zeta_i - \zeta_j\| : i \neq j \right\} > \varepsilon.$$

By $\Omega$ we denote the space of all real sequences $v = \left(v(j)\right)_{j=0}^{\infty}$ and $\left(\mathfrak{E}, \|\cdot\|\right)$ represent subspace of $\Omega$ and a Banach space. For a unit sphere $S(\mathfrak{E})$ and closed unit ball $B(\mathfrak{E})$, we call a sequence $(v_n) \subset \mathfrak{E}$ is said to be $\varepsilon$-separated sequence for some $\varepsilon > 0$, if separation of sequence $(v_n)$ denoted by $\operatorname{sep}(v_n) = \inf\left\{ \|v_n - v_m\| : n \neq m \right\} > \varepsilon$ [20].

In [15], the Opial property has been studied and was further studied in [19] and a powerful tool in deriving weak or strong convergence of iterative sequences is due to Opial. We call a Banach space $\mathfrak{E}$ to attain the Opial property, if for every weakly null sequence $(v_n) \subset \mathfrak{E}$ and every non-zero $v \in \mathfrak{E}$, we have

$$\liminf_{n\to\infty} \|v_n\| < \liminf_{n\to\infty} \|v_n + v\|.$$

As in [3], a Banach sequence lattice $\mathfrak{E}$ attains Fatou property, if for any $y \in \Omega$ and sequence $(y_n) \subset \mathfrak{E}_+$ with

$$\mathfrak{E}_+ = \{y \in \mathfrak{E} : y \geq 0\}$$

satisfying $0 \leq y_n(j) \nearrow y(j)$, that is, $y_n(j)$ increases to $y(j)$ as $n \to \infty$ for each $j \in \mathbb{N}$ and $\sup_n \|y_n\| < \infty$, then, $y \in \mathfrak{E}$, and $\|y\|_{\mathfrak{E}} = \lim_{n\to\infty} \|y_n\|_{\mathfrak{E}}$. Let $\mathfrak{E}$ be a real vector space, then we call $\tau : \mathfrak{E} \to [0, \infty]$ a modular if it satisfies:

$\psi(u) = 0$ if and only if $u = 0$.

$\tau(\kappa u) = \tau(u)$ for all $\kappa \in \mathcal{F}$ with $|\kappa| = 1$.

$\tau(\kappa u + \lambda v) \leq \tau(u) + \tau(v)$ for all $u, v \in \mathfrak{E}$ and $\kappa, \lambda \geq 0$ with $\kappa + \lambda = 1$.

Moreover, the modular $\tau$ is said to be convex if

$$\tau(\kappa u + \lambda v) \leq \kappa\tau(u) + \lambda\tau(v)$$

for all $u, v \in \mathfrak{E}$ and $\kappa, \lambda \geq 0$ with $\kappa + \lambda = 1$.

For any modular $\tau$ on $\mathfrak{E}$, the space

$$\mathfrak{E}_p = \{u \in \mathfrak{E} : \tau(\kappa u) < \infty, \text{for some } \kappa > 0\}$$

is called the modular space.

We say a modular $\tau$ satisfies $\delta_2$-condition ($\tau \in \delta_2$) if for any $\varepsilon > 0$, there exists constants $A \geq 2$ and $B > 0$ such that

$$\tau(2v) \leq A\tau(v) + \varepsilon$$

for all $v \in \mathfrak{E}_p$ with $\tau(v) \leq B$.

Also, we call $\tau$ to satisfy strong $\delta_2$-condition ($\tau \in \delta_2^s$) if $\tau$ satisfies $\delta_2$-condition for all $B > 0$ with $A \geq$ 2 dependent on $A$.

Throughout the text, we use the following notions:

$$v\#_r = (v(1), v(2), \cdots, v(r), 0, 0, \cdots) - \textit{called as truncation of } v \textit{ at } r,$$

$$v\#_{\mathbb{N}-r} = (0, 0, \cdots, 0, v(r+1), v(r+2), \cdots) - \textit{called as truncation of } v \textit{ at } r,$$

$$v\#_I = \left\{v = (v(r))_{r=1}^{\infty} : v(r) \neq 0 \textit{ for all } r \subseteq \mathbb{N} \textit{ and } v(r) = 0 \textit{ for all } r \in \mathbb{N} \setminus I\right\},$$

$$supp\, v = \{r \in \mathbb{N} : v(r) \neq 0\}$$

and $cl\mathfrak{E}$ represents closure of a set $\mathfrak{E}$.

Also, by $p = (p_j)$ we represent the bounded sequence with $p_j > 1$ for all $j \in N$.

## 2. The space $\mathcal{L}_s^{\vartheta}(p, \Delta_g^w)$

In this section, we introduce the space $\mathcal{L}_s^{\vartheta}(p, \Delta_g^w)$ and show that it attains uniform Opial property, paranormed structure and some other structures as well.

Following authors as in [1, 22–23], we define the space $\mathcal{L}_s^{\vartheta}(p, \Delta_g^w)$ as follows:

$$\mathcal{L}_s^{\vartheta}(p, \Delta_g^w) = \left\{v \in \Omega : \tau_{\Delta_g^w}(\gamma v) < \infty \textit{ for some } \gamma > 0\right\},$$

where $p = (p_i)$ is a sequence of positive real numbers with $p_i \geq 1$ for all $i \in \mathbb{N}$, $g = (g_i)$ is a sequence such that $g_i \neq 0$ for all $i \in \mathbb{N}$ and $s > 0$ equipped with Luxemburg norm

$$\|v\| = \inf\left\{\gamma > 0 : \tau_{\Delta_g^w}\left(\frac{v}{\gamma}\right) \leq 1\right\},$$

where

$$\tau_{\Delta_g^w}(v) = \sum_{r=1}^{w} |g(r)v(r)| + \sum_{i=1}^{\infty}\left(\frac{1}{h_i}\sum_{k \in \mathfrak{I}_i} k^{-s}\left|\Delta_g^w v(k)\right|\right)^{p_i}.$$

**Theorem 2.1** The functional $\tau_{\Delta_g^w}$ on $\mathcal{L}_s^{\vartheta}(p, \Delta_g^w)$ is convex modular.

Proof. Since $g_i \neq 0$ for all $i \in \mathbb{N}$, we see

$$\begin{aligned}
\tau_{\Delta_g^w} = 0 &\Leftrightarrow \sum_{r=0}^{w} |g(r)u(r)| + \sum_{k=0}^{\infty}\left(\frac{1}{h_k}\sum_{j \in \mathfrak{I}_k} j^{-s}\left|\Delta_g^w u(j)\right|\right)^{p_k} = 0 \\
&\Leftrightarrow \sum_{r=0}^{w} |g(r)u(r)| = 0 \text{ and } \sum_{k=0}^{\infty}\left(\frac{1}{h_k}\sum_{j \in \mathfrak{I}_k} j^{-s}\left|\Delta_g^w u(j)\right|\right)^{p_k} = 0 \\
&\Leftrightarrow u(r) = 0 \text{ for } r = 0,1,2,\ldots,w \ \& \Delta_g^w u(j) = 0 \ \forall j \in \mathfrak{I}_k, k \in \mathbb{N} \\
&\Leftrightarrow u = 0.
\end{aligned}$$

It is trivial that $\tau_{\Delta_g^w}(\beta u)=\tau_{\Delta_g^w}(u)$ for all scalars $\beta$ with $|\beta|=1$.

Using linearity of $\Delta_g^w$ and convexity of map $\tau \to |\tau|^{p_i}$, and for $u,v \in \mathcal{L}_s^{\vartheta}\left(p,\Delta_g^w\right)$ with $a \ge 0$, $b \ge 0$ and $a + b = 1$, we have

$$\begin{aligned}
\tau_{\Delta_g^w}(au+bv) &= \sum_{j=0}^{w}|g(j)(au(j)+bv(j)|+\sum_{k=0}^{\infty}\left(\frac{1}{h_k}\sum_{j\in\Im_k}\left|a\Delta_g^w u(j)+b\Delta_g^w v(j)\right|\right)^{p_k}\\
&\le \sum_{j=0}^{w}(|(ag(j)u(j)|+|bg(j)v(j)|)+\sum_{k=0}^{\infty}\left(\frac{1}{h_k}\sum_{j\in\Im_k}\left|a\Delta_g^w u(j)\right|+\left|b\Delta_g^w v(j)\right|\right)^{p_k}\\
&\le a\left[\sum_{j=0}^{w}|g(j)u(j)|+\sum_{k=0}^{\infty}\left(\frac{1}{h_k}\sum_{j\in\Im_k}\left|\Delta_g^w u(j)\right|\right)^{p_k}\right]+b\left[\sum_{j=0}^{w}|g(j)v(j)|+\sum_{k=0}^{\infty}\left(\frac{1}{h_k}\sum_{j\in\Im_k}\left|\Delta_g^w v(j)\right|\right)^{p_k}\right]\\
&= a\tau_{\Delta_g^w}(u)+b\tau_{\Delta_g^w}(v).
\end{aligned}$$

This shows that $\tau_{\Delta_g^w}$ is a convex modular on $\mathcal{L}_s^{\vartheta}\left(p,\Delta_g^w\right)$. ◊

We state the following important results Theorem 2.2 and Theorem 2.3 with out proof as are direct consequences of Theorem 2.1:

**Theorem 2.2** For $u \in \mathcal{L}_s^{\vartheta}\left(p,\Delta_g^w\right)$, the modular $\tau_{\Delta_g^w}$ on $\mathcal{L}_s^{\vartheta}\left(p,\Delta_g^w\right)$ satisfies the following properties:

($i$) if $0<\beta<1$, then $\beta^w\tau_{\Delta_g^w}\left(\frac{u}{\beta}\right)\le\tau_{\Delta_g^w}(u)$ and $\tau_{\Delta_g^w}\left(\beta u\right)\le\beta\tau_{\Delta_g^w}(u)$.

($ii$) if $\beta>1$, then $\tau_{\Delta_g^w}\left(u\right)\le\beta^w\tau_{\Delta_g^w}\left(\frac{u}{\beta}\right)$.

($iii$) if $\beta\ge1$, then $\tau_{\Delta_g^w}\left(u\right)\le\beta\tau_{\Delta_g^w}\left(u\right)\le\tau_{\Delta_g^w}\left(\beta u\right)$.

($iv$) if $\|u\|<1$, then $\tau_{\Delta_g^w}\left(u\right)\le\|u\|$.

($v$) if $\|u\|>1$, then $\tau_{\Delta_g^w}\left(u\right)\ge\|u\|$.

($vi$) if $\|u\|=1$, then $\tau_{\Delta_g^w}\left(u\right)=1$.

**Theorem 2.3** For any $u,v \in \mathcal{L}_s^{\vartheta}\left(p,\Delta_g^w\right)$, if $\tau_{\Delta_g^w}\in\Delta_2^s$, then for any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|\tau_{\Delta_g^w}(u+v)-\tau_{\Delta_g^w}(u)\right|<\varepsilon,$$

with $\tau_{\Delta_g^w}\left(u\right)\le L$ and $\tau_{\Delta_g^w}\left(v\right)\le\delta$.

**Theorem 2.4**

($i$) If $\tau_{\Delta_g^w}\in\Delta_2^s$, then for any $u \in \mathcal{L}_s^{\vartheta}\left(p,\Delta_g^w\right)$, $\|u\|=1$ if and only if $\tau_{\Delta_g^w}=0$.

($ii$) If $\tau_{\Delta_g^w}\in\Delta_2^s$, then for any $(u_n)\in\mathcal{L}_s^{\vartheta}\left(p,\Delta_g^w\right)$, $\|u_n\|\to0$ if and only if $\tau_{\Delta_g^w}\to0$.

($iii$) If $\tau_{\Delta_g^w}\in\Delta_2^s$, then for any $\delta=\delta(\varepsilon)>0$ such that $\|u\|\ge1+\delta$ whenever $\tau_{\Delta_g^w}\ge1+\varepsilon$.

**Theorem 2.5** For $\mathfrak{M} = max(1, \mathfrak{H} = sup_i p_i)$, the space $\mathcal{L}_s^{\vartheta}\left(p, \Delta_g^w\right)$ is complete paranormed space (not necessarily total paranormed) with

$$\mathfrak{G}_{\Delta_g^w}(u) = \sum_{j=0}^{w} |g(j)u(j)| + \left(\sum_{i=0}^{\infty}\left(\frac{1}{h_i}\sum_{k\in\mathfrak{I}_i} k^{-s}\left|\tau_{\Delta_g^w} u(k)\right|\right)^{p_k}\right)^{\frac{1}{\mathfrak{M}}}.$$

*Proof.* Using classical techniques, it can be proved and hence is omitted.

**Theorem 2.6** The space $\mathcal{L}_s^{\vartheta}\left(p, \Delta_g^w\right)$ attains Fatou property.

For $j \in \mathbb{N}$, we suppose that $u_j \in \mathcal{L}_s^{\vartheta}\left(p, \Delta_g^w\right)$ and let $\mathcal{H} = \sup_j \|u_j\| < \infty$ and $0 \le u_j(i) \nearrow u(i)$ as $j \to \infty$ for each $i \in \mathbb{N}$. Set $\mathfrak{B} = \sup_n \|u_n\|$, and since $\|u_n\| \le \mathfrak{B} < \infty$ for $n \in \mathbb{N}$, so that $0 \le \frac{u_n}{\mathfrak{B}} \le \frac{u_n}{\|u_n\|}$. Thus, $\tau_{\Delta_g^w} \frac{u_n}{\|u_n\|} \le 1$ and since $\tau_{\Delta_g^w}$ is monotone, we get

$$\tau_{\Delta_g^w}\left(\frac{u_n}{\mathfrak{B}}\right) \le \tau_{\Delta_g^w}\left(\frac{u_n}{\|u_n\|}\right) \le 1.$$

Employing the Beppo Levi theorem (see, [18]) and the fact that $\mathfrak{B}^{-1}u_n \to \mathfrak{B}^{-1}u$ as $n \to \infty$, we see that

$$\tau_{\Delta_g^w}\left(\frac{u}{\mathfrak{B}}\right) = \lim_{n\to\infty} \tau_{\Delta_g^w}\left(\frac{u_n}{\mathfrak{B}}\right) = \sup_n \tau_{\Delta_g^w}\left(\frac{u_n}{\mathfrak{B}}\right) \le 1.$$

This shows that $\|u\| \le \mathfrak{B}$ and $(\|u_n\|)$ is non-decreasing, so we have $\|u_n\| \to \mathfrak{B} = \sup_n \|u_n\|$ as $n \to \infty$. Now, we have by using norm definition that

$$\begin{aligned}
\|u_n\| &= \inf\left\{\lambda > 0 : \tau_{\Delta_g^w}\left(\frac{u_n}{\lambda}\right) \le 1\right\} \\
&= \inf\left\{\lambda > 0 : \sum_{j=0}^{w}\left|\frac{u_n(j)}{\lambda}\right| + \sum_{k=0}^{\infty}\left(\frac{1}{h_k}\sum_{j\in\mathfrak{I}_k} k^{-s}\left|\frac{\Delta_g^w u_n(j)}{\lambda}\right|\right)^{p_k} \le 1\right\} \\
&\le \inf\left\{\lambda > 0 : \sum_{j=0}^{w}\left|\frac{u(j)}{\lambda}\right| + \sum_{k=0}^{\infty}\left(\frac{1}{h_k}\sum_{j\in\mathfrak{I}_k} k^{-s}\left|\frac{\Delta_g^w u(j)}{\lambda}\right|\right)^{p_k} \le 1\right\} \\
&= \inf\left\{\lambda > 0 : \tau_{\Delta_g^w}\left(\frac{u_n}{\lambda}\right) \le 1\right\} = \|u\|.
\end{aligned}$$

Consequently, we see that $\sup_n \|u_n\| \le \|u\|$. Hence, we conclude that $\|u\| = \sup_n \|u_n\| = \lim_{n\to\infty} \|u_n\|$.

**Theorem 2.7** If $limsup_r p_r < \infty$, then the space $\mathcal{L}_s^{\vartheta}\left(p, \Delta_g^w\right)$ has uniform Opial property.

*proof.* Let $\varepsilon > 0$ be any arbitrary number and $u \in \mathcal{L}_s^{\vartheta}\left(p, \Delta_g^w\right)$ with $\|u\| \ge \varepsilon$. Let $(u_n)$ be any weakly null sequence in $\mathcal{S}\left(\mathcal{L}_s^{\vartheta}\left(p, \Delta_g^w\right)\right)$.

Since, $limsup_r p_r < \infty$, that is, $\tau_{\Delta_g^w} \in \delta_2^s$, by Theorem 2.4(ii), for each $\varepsilon > 0$, there is a $\delta \in (0,1)$ such that for each $u \in \mathcal{L}_s^{\vartheta}\left(p, \Delta_g^w\right)$ we have $\tau_{\Delta_g^w}(u) \ge \delta$.

Again, since $\tau_{\Delta_g^w} \in \delta_2^s$, by Theorem 2.3 for any $\varepsilon > 0$, there is a $\delta_1 \in (0, \delta)$ such that

$$\left|\tau_{\Delta_g^w}(x+y)-\tau_{\Delta_g^w}(x)\right|<\frac{\delta}{4}, \tag{1}$$

whenever $\tau_{\Delta_g^w}(x)\leq 1$ and $\tau_{\Delta_g^w}(y)\leq \delta$ and $u,w\in \mathcal{L}_s^{\vartheta}\left(p,\Delta_g^w\right)$.

Since $\tau_{\Delta_g^w}(u)<\infty$, so there exists a natural number $\mathfrak{J}_0$ such that

$$\sum_{j=0}^{w}\left|g(j)u(j)\right|+\sum_{i=\mathfrak{J}_0}^{\infty}\left(\frac{1}{h_i}\sum_{k\in\mathfrak{J}_i}k^{-s}\left|\Delta_g^w u(k)\right|\right)^{p_i}\leq\frac{\delta_1}{4}. \tag{2}$$

From (2), it follows that

$$\begin{aligned}\delta &\leq \sum_{j=0}^{w}\left|g(j)u(j)\right|+\sum_{i=0}^{\mathfrak{J}_0}\left(\frac{1}{h_i}\sum_{k\in\mathfrak{J}_i}k^{-s}\left|\Delta_g^w u(k)\right|\right)^{p_i}+\sum_{i=\mathfrak{J}_0+1}^{\infty}\left(\frac{1}{h_i}\sum_{k\in\mathfrak{J}_i}k^{-s}\left|\Delta_g^w u(k)\right|\right)^{p_i}\\ &\leq \sum_{i=0}^{\mathfrak{J}_0}\left(\frac{1}{h_i}\sum_{k\in\mathfrak{J}_i}k^{-s}\left|\Delta_g^w u(k)\right|\right)^{p_i}+\frac{\delta_1}{4}\end{aligned}$$

giving

$$\sum_{i=0}^{\mathfrak{J}_0}\left(\frac{1}{h_i}\sum_{k\in\mathfrak{J}_i}k^{-s}\left|\Delta_g^w u(k)\right|\right)^{p_i}\geq \delta-\frac{\delta_1}{4}>\delta-\frac{\delta}{4}=\frac{3\delta}{4}. \tag{3}$$

By the linearity of the operator $\Delta_g^w$ and weak convergence implies coordinatewise convergence, that is, $u_n\to 0$ weakly implies $u_n(i)\to 0$ for each $i\in\mathbb{N}$,\$ so there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$, we get

$$\sum_{i=0}^{\mathfrak{J}_0}\left(\frac{1}{h_i}\sum_{k\in\mathfrak{J}_i}k^{-s}\left|\Delta_g^w u_n(k)+\Delta_g^w u(k)\right|\right)^{p_i}>\frac{3\delta}{4}. \tag{4}$$

Again, using the fact that $u_n^w\to 0$, we can choose $\mathfrak{J}_0$ such that

$$\tau_{\Delta_g^w}\left(u\big|_{\mathfrak{J}_0}\right)\to 0\, as\, n\to\infty.$$

So, there exists a $n_1>n_0$ such that

$$\tau_{\Delta_g^w}\left(u\big|_{\mathfrak{J}_0}\right)\leq\delta_1 \text{ for all } n\geq n_1.$$

Because $(u_n)\in S\left(\mathcal{L}_s^{\vartheta}\left(p,\Delta_g^w\right)\right)$, that is, $\|u\|=1$, so by Theorem 2.3(i), we have $\tau_{\Delta_g^w}\left(u\big|_{\mathfrak{J}_0}\right)=1$. This implies that there exists $\mathfrak{J}_0$ such that

$$\tau_{\Delta_g^w}\left(u\big|_{\mathbb{N}-\mathfrak{J}_0}\right)\leq 1.$$

Now choose $\mathfrak{v}=u_n\big|_{\mathbb{N}-\mathfrak{J}_0}$ and $\mathfrak{w}=u_n\big|_{\mathfrak{J}_0}$. Then, $\tau_{\Delta_g^w}(\mathfrak{v})\leq 1$ and $\tau_{\Delta_g^w}(\mathfrak{w})\leq\delta_1$ for $\mathfrak{v},\mathfrak{w}\in\mathcal{L}_s^{\vartheta}\left(p,\Delta_g^w\right)$. So from equation (1), for all $n\geq n_1$, we have

$$\left|\tau_{\Delta_g^w}\left(u_n\big|_{\mathbb{N}-\mathfrak{J}_0}+u_n\big|_{\mathfrak{J}_0}\right)-\tau_{\Delta_g^w}\left(u_n\big|_{\mathbb{N}-\mathfrak{J}_0}\right)\right|<\frac{\delta}{4},$$

which implies that

$$\tau_{\Delta_g^w}(u_n)-\frac{\delta}{4}<\tau_{\Delta_g^w}\left(u_n\big|_{\mathbb{N}-\mathfrak{J}_0}\right) \textit{ for all } n\geq n_1.$$

That is,

$$\sum_{i=0}^{w}\left|u_n(i)\right|+\sum_{i=\Im_0+1}^{\infty}\left(\frac{1}{h_i}\sum_{k\in\Im_i}k^{-s}\left|\Delta_g^w u_n(k)\right|\right)^{p_i}>1-\frac{\delta}{4}\ \textit{for all}\ n\geq n_1. \tag{5}$$

Again, since $\tau_{\Delta_g^w}\left(u_n\big|_{\mathbb{N}-\Im_0}\right)\leq 1$ and $\tau_{\Delta_g^w}\left(u_n\big|_{\mathbb{N}-\Im_0}\right)\leq\frac{\delta_1}{4}<\delta_1$, so from (1), we have

$$\left|\tau_{\Delta_g^w}\left(u_n\big|_{\mathbb{N}-\Im_0}+u\big|_{\mathbb{N}-\Im_0}\right)-\tau_{\Delta_g^w}\left(u_n\big|_{\mathbb{N}-\Im_0}\right)\right|<\frac{\delta}{4}$$

which shows that

$$\left|\tau_{\Delta_g^w}\left(u_n\big|_{\mathbb{N}-\Im_0}+u\big|_{\mathbb{N}-\Im_0}\right)>\tau_{\Delta_g^w}\left(u_n\big|_{\mathbb{N}-\Im_0}\right)\right|-\frac{\delta}{4}. \tag{6}$$

Now, from equations (4–6) and the linearity property of the operator $\Delta_g^w$, we have

$$\begin{aligned}\tau_{\Delta_g^w}\left(u_n+u\right)&=\sum_{j=0}^{w}\left|u_n(j)+u(j)\right|+\sum_{i=0}^{\Im_0}\left(\frac{1}{h_i}\sum_{k\in\Im_i}k^{-s}\left|\Delta_g^w u_n(k)+\Delta_g^w u(k)\right|\right)^{p_i}\\&+\sum_{i=\Im_0+1}^{\infty}\left(\frac{1}{h_i}\sum_{k\in\Im_i}k^{-s}\left|\Delta_g^w u_n(k)+\Delta_g^w u(k)\right|\right)^{p_i}\\&>\sum_{i=0}^{\Im_0}\left(\frac{1}{h_i}\sum_{k\in\Im_i}k^{-s}\left|\Delta_g^w u_n(k)+\Delta_g^w u(k)\right|\right)^{p_i}+\tau_{\Delta_g^w}\left(u_n\big|_{\mathbb{N}-\Im_0}\right)-\frac{\delta}{4}\\&>\frac{3\delta}{4}+\left(1-\frac{\delta}{4}\right)-\frac{\delta}{4}=1+\frac{\delta}{4}.\end{aligned}$$

Since, $\tau_{\Delta_g^w}\in\delta_2^s$, so by Lemma 2.3(ii), there is a $\lambda>0$ such that $\|u_n-u\|\geq 1+\lambda$, hence for $n\to\infty$ that

$$\liminf\|u_n+u\|\geq 1+\lambda.$$

Choosing different values of $m$, $s$, $g$, we have following deductions:

**Deduction 2.8** Choosing $s = 0$ and $g = 1$, the space $\mathcal{L}_s^{\vartheta}\left(p,\Delta_g^w\right)$ then reduced to what has been introduced in [11].

**Deduction 2.9** Choosing $s = 0$, $w = 0$, $g = 1$ and $\vartheta = (2^r)$, the space $\mathcal{L}_s^{\vartheta}\left(p,\Delta_g^w\right)$ then reduced to what has been introduced in [22].

**Deduction 2.10** Choosing $w = 0$ and $g = 0$, the space $\mathcal{L}_s^{\vartheta}\left(p,\Delta_g^w\right)$ then reduced to what has been introduced in [16].

**Deduction 2.11** Choosing $s = 0$, $w = 1$, $g = 0$, $p_r = p$ and $\vartheta = (2^r)$ for all $r\in\mathbb{N}$, the space $\mathcal{L}_s^{\vartheta}\left(p,\Delta_g^w\right)$ then reduced to what has been introduced in [21].

### Acknowledgment

The authors are grateful to the reviewers for their meticulous reading and suggestions which improved the presentation of the paper.

### Funding

The authors extend their appreciation to the Deanship of Scientific Research at Saudi Electronic University for funding this research (8367).

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