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Everywhere divergent extended Hermite - Fejer interpolation process

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Abstract

The purpose of this paper is to construct a polynomial of different degree with some conditions in the interval $I=[-1,1]$ by using Hermite - Fejer interpolation polynomial (HFI) of degree at most $(4n-1)$ that agree with $f \in C[-1,1]$ and has zero derivative at each nodes. Also, we investigate all extensions of (HFI) on $(-1,1)$ which are divergent everywhere.

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1. Introduction

Let us begin with the following triangular matrix of numbers

$$M = \{x_{kn}\}_{k=1}^{n} \, n = 1,2,3,\ldots \tag{1}$$

Such that

$$-1 \le x_{1n} \le x_{2n} \le x_{3n} \le \ldots \le x_{kn} \le 1 \tag{2}$$

By $C(I)$ we denote the set of all functions $f(x)$ that are continuous on $I=[-1,1]$. We further denote by $H_n(f,M,x) \equiv H_n(f,M) \equiv H_n(M,x)$ polynomial of degree $\le 4n-1$ uniquely determined from the following conditions.

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$$H_n(f,M,x_{kn}) = f(x_{kn}); H_n^{(r)}(f,M,x_{kn}) = 0, k = 1,2,\ldots, r = 1,2,3. \quad (3)$$

By adding the points ∓ 1 to *M*, we can construct another interpolation processes $H_{In}(f,M,x)$, $I = 1,2,\ldots,256.$ if we add the points ∓ 1 to *M*. These are quite naturally called the extensions of the step parabol as of higher. The processes $\{H_n(f,M,x)\}_{n=1}^{\infty}$ is called Hermite−Fejere interpolation process (HFI). The interpolation polynomial can be written in the form $H_n(x) = \sum_{k=1}^{n} f(x_{kn})\left[1 - \frac{w_n''(x_{kn})}{w_n'(x_{kn})}(x - x_{kn})\right] I_{kn}^2(x)$ where the fundamental polynomial $I_k(x)$ of Lagrange and the polynomial $w_n(x)$ are defined by

$$w_n(x) = \sum_{k=1}^{n}(x - x_{kn}) \text{ and } I_{kn}(x) = \frac{w_n(x)}{w_n(x_{kn})(x - x_{kn})}$$

A generalization of Lagrange interpolation is provided by Hermite - Fejer interpolation polynomial that given a non-negative integer *k* and node *X* defined by 1 and 2, the polynomial

$H_{In}(f,X,x) = H_{In}(X,x)$ of *f* is the unique polynomial of degree at most $(I + 1)n - 1$, which satisfies the $(I+1)n$ conditions. $H_{In}(f,X,x_{kn}) = f(x_{kn}), 0 \le k \le n-1$

$H_{In}^{(r)}(f,X,x_{kn}) = 0, 1 \le r \le I, 0 \le k \le n-1$ L. Fejer [1] showed that if one takes the n-th row of the matrix $M = T$, as zeros of *T* chebychev polynomial of first kind

$$T = \{x_k\} = \left\{cos\frac{2k-1}{2n}\pi\right\}. k = 1,2,\ldots,n, n = 1,2,\ldots \quad (4)$$

Berman studied the process $\{H_n(M,x)\}$ for the case of nodes

$$x_0 = 1, x_k = cos\frac{2k-1}{2n}\pi, k = 1,2,\ldots,n, n = 1,2,\ldots, x_{n+1} = -1 \quad (5)$$

Obtained by adding the nodes ∓ 1 to the system in 4. In [2] it is shown that this process constructed for afunction $f(x) = |x|$ diverges at $x = 0$, while [3] it was shown that process constructed for $f(x) = x^2$ diverges everywhere in $I = (-1,1)$. Cook and Mils [4] in 1975, who showed that if $(x) = (1-x^2)^3$, then $H_{3n}(T_{\mp 1}, f, 0)$ divrges.

The result in [4] extended by my paper [5] that showed $H_{3n}(T_{\mp 1}, f, x)$ diverges at each points in $(-1,1)$. Byrne and Smith [6] investigate Berman's phenomenon in the set of $(0,1,2)$. In the last my paper [7], it showed that HFI interpolation polynomial of chebychev of the first kind converges for $f(x) = x^2$ in $(-1,1)$, while diverges for $f(x) = x$ for all $x \ne 0$ in $(-1,1)$ where n is an even integer number at the node of degree $4n+1$. In [8] Berman considers special case of the processes $\{H_n(f,M,x)\}$, when $\{x_{kn}\}_{k=1}^{n}$ are the roots of polynomials

$$P_n(x) = \frac{\sin\left(n + \frac{1}{2}\right)\theta}{sin\frac{\theta}{2}}, x = cos\theta \text{ and}$$
$$P = cos\frac{2k\pi}{2n+1}, k = 1,2,\ldots,n, n = 1,2,\ldots \quad (6)$$

The polynomial $P_n(x)$ are the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with parameters $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$. The following theorem is due to L. Fejer [1].

Theorem 1.1: *If* $\in C(I)$, *then* $\lim_{n\to\infty}\|H_n(f,T) - f\|_{\infty} = 0$ *where* $\|\cdot\|_{\infty}$ *denotes the uniform norm on* $C(I)$.

The first uniform error estimate for the rate convergence of (HFI) was proved by Popoviciu [9]. His estimate was framed in terms of modulus of continuity of *f* defined by

$$w(f,\delta) = sup\{ f(x) - f(y) : x, y \varepsilon I, |x - y| \le \delta \}$$

Normally he proved the following theorem

Theorem 1.2 (Popviciu): *If* $f(x) \in C(I)$*, then for* $n = 1,2,3,\ldots \|H_n(f,T) - f\|_\infty \le 2w\left(f, \frac{1}{\sqrt{n}}\right)$ *Above result was subsequently improved by Moldovan* [10] *and by Shisha and Mond* [11].

Let us dente by $H_n(f,M,x_{kn}) \equiv H_n(M,x_{kn}) \equiv H_n(f,M)$, the interpolation polynomial of degree $4n-1$, sometimes called Hermite - Fejer interpolation polynomial of higher order uniquely determined from the following set of conditions as in [3].

In this paper consider special case of the processes $\{H_n(f,X,x_k)\}$ for the case when the points $\{x_k\}$ are the roots of the Legendre polynomial $P_n(x)$. In this case

$$x_k = cos\frac{2k\pi}{2n+1}, k = 1,2,\ldots,n, n = 1.2,\ldots \tag{7}$$

We want to study the process $H_n(f,M,x)$ that are divergent in (–1,1) for the case of nodes

$$x_0 = 1, x_k = cos\frac{2k\pi}{2n+1}, x_{n+1} = -1, k = 1,2,\ldots,n, n = 1,2,\ldots$$

Obtained by extending the nodes of matrix 7 by adding ∓ 1.

2. The Main Result

Theorem 2.1: *The following Hermite – fejer interpolation polynomials (HFI) are divergent in (–1,1).*

1. $H_1(f,P,x_k)$ of degree at most $4n+2$ by th $(4n+3)$ condition.

$$H_1(f,P,1) = f(1), H_1^{(r)}(f,P,1) = 0, r = 1,2$$

2. $H_2(f,P,x_k)$ of degree at most $4n+3$ by $4n+4$ conditions.

$$H_2(f,P,\mp 1) = f(\mp 1), H_2^{(r)}(f,P,1) = 0, r = 1,2.$$

3. $H_3(f,P,x_k)$ of degree at most $4n + 3$ by the $4n + 4$ conditions.

$$H_3(f,P,1) = f(1), H_3^{(r)}(f,P,1) = 0, r = 1,2,3$$

4. $H_4(f,P,x_k)$ of degree at most $4n+5$ by the $4n+6$ conditions.

$$H_4(f,P,\mp 1) = f(\mp 1), H_4^{(r)}(f,P,\mp 1) = 0, r = 1,2.$$

5. $H_5(f,P,x_k)$ of degree at most $4n+7$ by the $4n+8$ conditions.

$$H_5(f,P,\mp 1) = f(\mp 1), H_5^{(r)}(f,P,\mp 1) = 0, r = 1,2,3$$

The theorem will be proved via a sequence of lemmas in the next section.

3. Technical Preliminaries

We shall quite frequently make use of the following lammas before proof the main theorem.

Lemma 3.1: *From the differential equation satisfied by $P_n(x)$, we obtain the following values:*

$$P_n'(1) = \frac{1}{2}n(n+1) = (-1)^{n-1}P_n'(-1)$$

$$P_n''(1) = \frac{1}{8}(n-1)n(n+1)(n+2) = -1)^n P_n''(-1)$$

$$P_n'''(1) = \frac{1}{48}n(n-1)(n-2)(n+1)(n+2)(n+3) = -1)^{n-1}P_n'''(-1)$$

$$P_n^{(iv)}(1) = \frac{1}{384}n(n-1)(n-2)(n-3)(n+1)(n+2)(n+3)(n+4) = -1)^n P_n^{(iv)}(-1)$$

$$P_n^4(x)]'_{x=\mp 1} = \mp 2n(n+1), P_n^4(x)]''_{x=\mp 1} = \frac{n(n+1)}{2}(7n^2+7n-2)$$

$$P_n^4(x)]'''_{x=\mp 1} = \frac{\mp 3n(n+1)}{4}(11n^4+22n^3-33n^2-26n+481)$$

Lemma 3.2: *The following estimates hold* [12].

$$(1-x^2)^{\frac{1}{4}}P_n(x) \le \frac{\sqrt{2}}{\pi}n^{-\frac{1}{2}}, |P_n(x)| \le 1$$

$$\left.\begin{aligned} &(1-x^2)^{\frac{3}{4}}|P_n'(x)| \le \frac{\sqrt{8}}{\pi}n^{\frac{1}{2}} \\ &(1-x^2)^{\frac{1}{2}}|P_n'(x)| \le n \\ &(1-x^2)|P_n(x)P_n'(x)| \le 2 \end{aligned}\right\} n > 3, x \in [-1,1]$$

We often make use of a simple property of uniform convergent (u.c) operators [10] which states that if $u_n(f,x)$ is a linear polynomial operator, then f u.c. on I, we have uniformly for $x \in I$

$$\frac{u_n^{(k)}(f,x)}{n^{2k}} = o(1) \tag{8}$$

Where $u_n^{(k)}(f,x)$ is the n-th derivative of $u_n(f,x)$

4. Proof of the main result (Theorem)

Our method of proof constitutes expressing the polynomial $H_i(f,M,x)$ in terms of the well- known Hermite – Fejer interpolation polynomial $H_n(f,M,x)$ and the reminder (error) function, say

$$E_i(f,M,x) = H_i(f,M,x) - H_n(f,M,x) \tag{9}$$

Thus for $H_i(f,M,x)$ to converge to *f*, the error function $E_i(f,M,x)$ must tends to zero as n approaches to infinity. It is worth – mention that the uniformly convergent extension $H_I(f,M,x)$ was study by Sharma and Tzimbalario [13]. This will form the basis of the proof of our theorem. $H_i(f,M,x)$ of degree at most $4n+1$ by the $4n+2$ conditions such that :

$$H_i(f,P,\mp 1) = f(\mp 1) \text{ and } H_i^{(r)}(f,M,x_k) = 0, r = 1,2,3 \tag{10}$$

As indicated earlier, we shall take $H_i(f,P,x)$ as our starting points

1. Consider, now the extension $H_1(f,P,x)$ and comparing the two, we find that

$$E_i(f,M,x) = H_i(f,M,x) - H_n(f,M,x) = (1-x)(a_1x+b_1)P_n^4(x)$$

Where a_1 and b_1 are constants independent of x to be determined from the following two conditions:

$$E_1'(F,P,x) = (a_1x + b_1)(1-x)P_n^4(x)]'_{x=1},$$

since

$$(1-x)P_n^4(x)]'_{x=1} = -1; (1-x)P_n^4(x)]''_{x=1} = -4n(n+1)$$

and

$$(1-x)P_n^4(x)]'''_{x=1} = -\frac{3}{4}n(n+1)(7n^2+7n-2),$$

then

$$a_1 = -4n(n+1)H_1'\ (f,P,1) + \frac{1}{2}H_1''\ (f,P,1)$$

and

$$b_1 = \frac{1}{2}(2+2n(2n+1))H_1'(f,P,1) - \frac{1}{2}H_1''(f,P,1)$$

From Lemmas 3.1, 3.2 and 8, we have $E_1(f,P,x)$ dose not tends to zero, which implies the divergent of $H_1(f,P,x)$ on $(-1,1)$.

2. The error function is given by

$$E_2(f,P,x) = H_i(f,P,x) - H_2(f,P,\mp 1) = (a_2x+b_2)(1-x^2)P_n^4(x) \tag{11}$$

$$E_2'(f,P,x) = (a_2x+b_2)(1-x^2)P_n^4(x)]'_{x=1}$$

We apply the additional condition to obtain the following equations:

$$-H_i'f,P,1) = -2(a_2+b_2) \Rightarrow -\frac{1}{2}H_i'(f,P,1) = -a_2 - b_2 \tag{12}$$

$$-H_i''(f,P,1) = -a_2[N+6] - b_2[N+2] \tag{13}$$

Solving Equations 12 and 13, we get

$$a_2 = \frac{1}{4}H_i''(f,P,1) + \frac{1}{8}H_i'(f,P,1)(-N+2), \text{ where } N = 2n(n+1)$$

The divergent of $H_2(f,P,x)$ is evident from equation 11, Lemma 3.2 and 8, that is

$$E_2(f,P,x) \nrightarrow 0 \text{ in } (-1,1)$$

3. $E_3(f,P,x) = (1-x)^2(a_3x+b_3)P_n^4(x)$

$$H_i'(f,P,x) - H_3'(f,P,x) = (a_3x+b_3)[(1-x^2)P_n^4(x)] + a_3(1-x)^2P_n^4(x)$$
$$H_i''(f,P,x) - H_3''(f,P,x) = (a_3x+b_3)[(1-x)^2\ P_n^4(x)]'' + 2a_3[(1-x)^2P_n^4(x)]'$$
$$H_i'''(f,P,x) - H_3'''(f,P,x) = (a_3x+b_3)[(1-x)^2\ P_n^4(x)]''' + 3a_3[(1-x)^2P_n^4(x)]''$$

Since $(1-x^2)P_n^4(x)]''_{x=1} = 2$ and $(1-x^2)P_n^4(x)]'''_{x=1} = 12n(n+1)$ and by apply

The additional conditions to obtain the following:

$$a_3 = 2n(n+1)H_i''(f,P,1) - \frac{1}{3}H_i'''(f,P,1)$$

and

$$b_3 = -\frac{1}{2}[2n(n+1)+1]H_i''(f,P,1) + \frac{1}{3}H_i'''(f,P,1)$$

as $n \to \infty$, $E_3(f,P,x) \not\to 0$ Hence, by using Lemma 3.1 and Equation 8 $H_3(f,P,x)$ Divergent

4. $E_4(f,P,x) = (1-x)^2(a_4x+b_4)P_n^4(x)$

$$E_4''(f,P,x) = H_i''(f,P,x) + H_4''(f,P,x)$$
$$= [(1-x^2)^2P_n^4(x)][a_4x+b_4]'' + 2[(1-x^2)^2P_n^4(x)]' + (a_4x+b_4)' + [(1-x^2)^2P_n^4(x)]''[a_4x+b_4]$$

$-H_i''(f,P,1) = 8(a_4+b_4)$ and $-H_i''(f,P,-1) = 8(-a_4+b_4)$

$$\left.\begin{aligned} a_4 &= -\frac{1}{16}H_i''(f,P,1) + H_i''(f,P,-1) \\ b_4 &= -\frac{1}{16}[H_i''(f,P,1) + H_i''(f,P,-1)] \end{aligned}\right\} \tag{14}$$

From Equations 11 and 14, by using Lamma 3.1 and Equation 8 the error function $E_4(f,P,x)$ dose not tend to zero.

5. $E_5(f,P,x) = (1-x^2)^2[(1-x)^3a_5 + (1-x)^2b_5 + (1-x)c_5 + d_5]P_n^4(x)$

By apply the additional conditions of $H_5(f,P,x)$, we have the following equations

$$-H_i''(f,P,+1) = 8d_5$$
$$-H_i'''(f,P,1) = -24c_5 + [24+48n(n+1)]d_5$$
$$-H_i'''(f,P,-1) = -[480+384n(n+1)]a_5 - 6[16n(n+1)+12]c_5 - 6[8n(n+1)+4]d_5$$

From the above system of equations, it is $d_5 = -\frac{1}{8}H_i''(f,P,1) \not\to 0$ as $n \to \infty$, that is, the error function dose not tends to zero as n approaches to infinity. Hence $H_5(f,P,x)$ is divergent.

It would be sometimes helpful to discuss uniform convergence of certain extensions by actual construction also. For instance $H_6(f,M,x)$ of degree almost $4n+3$ by the following conditions

$$H_6(f,M,\mp 1) = f(\mp 1);\ H_6^{(r)}(f,M,\mp 1) = 0,\ r = 1$$

Is seen to have the following form

$$H_6(f,M,x) = f(1)\lambda_0(x) + f(-1)\lambda_{n+1}(x) + \sum_{k=1}^{n} f(\zeta_k)[1 + A_1(x-\zeta_k)^+ A_2(x-\zeta_k)^2$$
$$+[1 + A_1(x-\zeta_k)^+ A_2(x-\zeta_k)^2 + A_3(x-\zeta_k)^3]\left[\frac{(1-x^2)^2}{(1-\zeta_k^2)^2} I_k^4(x)\right]$$

Where

$$A_1 = 0,\ A_2 = \frac{\zeta^2}{2(1-\zeta^2)^2} + \frac{2(n^2+n+1)}{3(1-\zeta_k^2)},\ A_3 = \frac{4\zeta_k^2}{3(1-\zeta_k^2)} + \frac{2(n^2+n+2)}{(1-\zeta_k^2)},$$

and

$$\lambda_0 = \lambda_{n+1}(-x) = \left(\frac{1+x}{2}\right)^2 P_n^4(x)[1+(2n^2+2n+1)(1-x)]$$

$I_k(x), k = 1, 2, \ldots, n$ is the fundamental Lagrange interpolation polynomial built on the node system M.

5. Conclusion

In this paper investigate all extensions of Hermite – fejer interpolation polynomials (HFI) on (–1,1) which are divergent everywhere. On considering special case of the processes $\{H_n(f, X, x_k)\}$ when the points $\{x_k\}$ are the roots of the Legendre polynomial w. Obtained by extending the nodes of matrix (7) by adding ∓ 1.

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