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Several generalized K-shadowing properties characteristics in metric space

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Abstract

We focus here on K-shadowing characteristic due to its significant mathematical aspects and application. Several of the format's common characteristics are demonstrated in this essay. If (W, d) is a metric space with dimensions $z,v:(W,d)\to(W,d)$ be mapping have the K-shadowing characteristic. We demonstrate the K-shadowing properties of the mappings $z\times v$ $z+v$, z^n and zov.

Keywords: Shadowing, K-shadowing, Metric space

1. Introduction

The discrete dynamical system's attraction patterns on an unlimited time interval have indeed been utilized to study the system's characteristics. Regarding complex structures like chaos theory, a mathematical model of convergences is required in order to do this. As a result, there are currently pseudo-orbits and behavior's that results simply mirrors the behaviors of approximation systems. As a result, the possibility of a real orbit close to a pseudo-orbit is clearly highlighted and given serious thought. The notion of shadowing characteristic seems to be the most effective approach to put these principles into action. In reality, the idea of shadowing is essential for comprehending the dynamical properties [1–3].

The pseudo-orbit shadowing feature is among the most central themes in system dynamics [4–5], that is specifically linked to stability analysis and chaotic, as shown in [6–7], and is integral aspect

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of stabilization and stochastic theories [8]. Out of a numeric standpoint, when h seems to have the pseudo-orbit shadowing characteristic, then perhaps the orbits generated by numerical calculations represent the genuine dynamic behavior of h. Recently developed the concept of the average-shadowing feature in the research of chaos dynamics [9–14].

The opposite concept, which states that each real orbit of the systems could be roughly approximate by a pseudo orbital containing particular features, is also significant. In reality, those fictitious orbits are selected from just a family of orbits produced by continuous mapping. Inverted shadow is a notion that was first presented in Koleden et al., [15, 16] utilizing the approach. Bi-shadowing, that was developed by Diamond et al. [17] is a mixture of the ideas of shadowing and inverted shadowing; which include [18]. Two scenarios involving bi-shadowing were examined: a first involved finite-dimensional systems in and the second involved infinitely systems in Anosov [19].

Regarding collection dynamics in a metric space including an applicability to iterated system, bi-shadowing was taken into consideration; view [20]. Diamond et al., [21] were investigated the fatuous homeomorphisms on compact configurations and displayed all inverse shadowing and bi-shadowing characteristics in relation to a class of techniques that are defined by continuous mappings from exhaust system into the space of bi-infinite sequences in the exhaust system also with product topology [22]. Such systems were shown to demonstrate the idea of bi-shadowing with regard to continuous comparative mappings for these differently under different circumstances. Al-Badarneh additionally provided instances of subcategories incorporating Reich maps, and Kannan maps to further show his results.

In addition to stating and proving several general statements and conclusions about the K-shadowing feature, this work provides many preliminary information that is necessary.

2. Preliminaries

Definition 2.1: [22] Assume the dynamical system on Y generated by the rounds of z, that is $z^0 = id_Y$ and $z^{k+1} = z^k \circ z$, for all $k \in \mathbb{N}$. Let $z : (W,d) \to (W,d)$ be a mapping defined on a metric space (W,d). We will link the dynamical system that corresponds to the map z. If $w_{k+1} = z(w_k)$ then the sequence $\{w_k\}_{k=0}^{\infty} \subset W$ is considered to be a genuine orbit of z. This holds true for every $k \in \mathbb{N}$. A γ pseudo-orbit of z is a series of the form $\{h_k\}_{k=0}^{\infty} \subset W$, where $d(h_{k+1}, z(h_k)) \leq \gamma$ for every $k \in \mathbb{N}$.

Definition 2.2: [23] A continuous map $z : W \to W$ is called bi-shadowing with respect to composition maps on W and with positive criteria ξ and λ for any γ-pseudo-orbit $\{\ell_n\}_{n=0}^{\infty}$ of z with $0 \leq \gamma \leq \lambda$ and any $\sigma \in C(W)$ satisfying $\gamma + \sup_{w \in W} d(\sigma(w), z(w)) \leq \lambda$ then exist a true orbit $\{w_n\}_{n=0}^{\infty}$ of z such that $d(\ell_n, w_n) \leq \xi(\gamma + \sup_{w \in W} d(\sigma(w), z(w))) \forall n \in \mathbb{N}$.

3. The Most Important Results

In this part, we present concept of K-shadowing and demonstrate the key findings involving mappings with the characteristic of K-shadowing as in metric spaces (W,d) and (W^*, d').

Definition 3.1: According to a comparative category of mappings $P(W)$ made up of continuous mappings on W and with positive criteria ξ and λ, a continuous mapping $z : W \to W$ is said to be K-shadowing if, for just every provided $\gamma - \inf_{w \in W} d(\sigma(w), z(w)) \leq 2\lambda$ with $0 \leq \gamma \leq \lambda$, $\sigma \in P(W)$ with and γ-pseudoorbit $\{w_k\}_{k=0}^{\infty}$ of σ then, $\forall k \in \mathbb{N}$, $d(h_k, w_k) \leq \xi(2\gamma - \inf_{w \in W} d(\sigma(w), z(w)))$.

Theorem 3.2: If $z, \upsilon : W \to W$ are maps with the K-shadowing property, then $z \circ \upsilon$ has the K-shadowing property.

Proof: If z has K-shadowing, then every chosen γ'-pseudo-orbit $\{w_k\}_{k=0}^{\infty}$ of z with $0 \le \gamma' \le \lambda'$ and any $\gamma \in C(W)$ fulfil:

$$\gamma' - \inf_{w \in W} d(\sigma(w), z(w)) \le \lambda'$$

There is a real orbit of $\{\xi_k\}_{k=0}^{\infty}$ of η with the properties: $d(w_n, \xi_n) \le \xi'(\gamma' - \inf_{w \in W} d(\sigma(w), z(w)))\ \forall k \in \mathbb{N}$.

Because v has k-shadowing for every chosen γ''-pseudo-orbit $\{h_k\}_{k=0}^{\infty}$ of h with $0 \le \gamma'' \le \lambda''$, as well as any fulfilling $\sigma \in C(W)$:

$$\gamma'' - \inf_{w \in W} d(\sigma(w), z(w)) \le \lambda''$$

it has a real orbit of $\{\psi_k\}_{k=0}^{\infty}$ of σ such that,

$$d(h_k, \psi_k) \le \xi''(\gamma'' - \inf_{w \in W} d(\sigma(w), z(w)))$$

Presently, for such $\zeta, \zeta' \ge 1$, set $\xi = \zeta\zeta'\xi'''$ whenever $\xi''' = \max\{\lambda', \lambda''\}, \lambda = \min\{\xi', \xi''\}, \lambda \ge \gamma \ge \max\left\{\frac{\gamma' + \gamma''}{\zeta'}, \frac{2\zeta\lambda + \gamma'}{\zeta}, \frac{2\zeta\lambda + \gamma''}{\zeta}\right\} \ge 0$, for the a specific γ-pseudo-orbit $\{l_k\}_{k=0}^{\infty}$ of $z \circ \upsilon$, and $\omega = \eta \circ \sigma \in C(W)$ fulfil:

$$\gamma - \inf_{w \in W} d(\sigma(w), z \circ \upsilon(w) \le \lambda$$

$$\inf_{w \in W} d(\eta(\sigma(w)), \upsilon(\mathrm{z}(w)) \le \lambda - \gamma$$

$$\inf_{w \in W} d(\sigma(w), z(w)) \le 2\zeta(\lambda - \gamma) \le \zeta\lambda - 4\frac{\zeta^2\lambda + \gamma'}{\zeta} \le \lambda - \gamma' \le \lambda' - \gamma'$$

So, a genuine orbit of η with the properties: $\{\xi_k\}_{k=0}^{\infty}$ occurs.

$$d(w_k, \xi_k) \le \xi'(\gamma' - \inf_{w \in W} d(\sigma(w), z(w))),\ \forall k \in \mathbb{N}$$

Similar to that, we may obtain:

$$\inf_{w \in W} d(\sigma(w), z(w)) 2\zeta(\lambda - \gamma) \le \zeta\lambda - 4\frac{\zeta^2\lambda + \gamma''}{\zeta} \le \lambda - \gamma'' \le \lambda'' - \gamma''$$

There is genuine orbit of $\{\psi_k\}_{k=0}^{\infty}$ of σ , and

$$d(h_k, \psi_k) \le \xi''(\gamma'' - \inf_{w \in W} d(\sigma(w), z(w)))_0^{\infty}\ \forall k \in \mathbb{N}.$$

Therefore, a genuine orbit of ρ with the properties $\{l'_n\}_{n=0}^{\infty} = \{\sigma(\eta(l_k))\}_{k=0}^{\infty}$ exists. It has the following properties:

$$\begin{aligned}
\frac{1}{\zeta} d(l_k, l'_k) &\le d(w_k, \xi_k) + d(h_k, \psi_k) \le \xi'(\gamma' - \inf_{w \in W} d(\sigma(w), z(w))) \\
&+ \xi''(\gamma'' - \inf_{w \in W} d(\sigma(w), \upsilon(w))) \le \xi'''(\gamma' - \inf_{w \in W} d(\eta(w), z(w))) \\
&+ \xi'''(\gamma'' - \inf_{w \in W} d(\sigma(w), \upsilon(w))) \le \xi'''(\gamma' + \gamma'' - \inf_{w \in W} d(\eta(w), z(w)) - \inf_{w \in W} d(\sigma(w), \upsilon(w))) \\
&\le a'''\left(\delta' + \delta'' + r' \sup_{y \in Y} d(\theta(\varnothing(y)), h(g(y)))\right) \le \zeta'\xi'''\left(\frac{\gamma' + \gamma''}{\zeta'} - \inf_{w \in W} d(\eta(\sigma(w)), \upsilon(\mathrm{z}(w))\right)
\end{aligned}$$

Thus,

$$d(l_n, l'_n) \le \zeta\zeta'\xi'''\left(\frac{\gamma'+\gamma''}{\zeta'} - \inf_{w\in W} d(\eta(\sigma(w)), \upsilon(\mathrm{z}(w))\right) \le \xi(\gamma - \inf_{w\in W} d(\eta(\sigma(w)), \upsilon(\mathrm{z}(w))) \forall k \in \mathbb{N}$$

Therefore, $z \circ \upsilon$ has the K-shadowing property.

Proposition 3.3: Suppose $z: W \to W$ be a mapping. About every $k \in \mathbb{N}$, if z has the K-shadowing characteristic. Thus, z^k has K-shadowing characteristic.

Proof: Using Theorem 3.2's Induction Principle, we may demonstrate this outcome. Suppose (W, d') and (H, d'') be metric space, $z: W \to W$ and $\upsilon: H \to H$ be mappings with $d((w,h),(w',h')) = d'(w,w') + d''(h,h')$. Including all $w \in W$ and for all $h \in H$, we constructed the mapping $(z \times \upsilon)(w,h) = (z(w), \upsilon(h))$. to demonstrate that its metric space $(W \times H, d)$ exists.

Assume that $(w,h),(w',h')$ and $(w'',h'') \in W \times H$.

1. Due to the fact that $d((w,h),(w',h')) = d'(w,w') + d''(h,h')$. Additionally, there is $d'(w,w') \ge 0$ and $d''(h,h') \ge 0$. Consequently, $d((y,z),(y',z')) \ge 0$.
2. $d((w,h),(w',h'))$, iff $d'(w,w') + d''(h,h') = 0$,
   iff $d'(w,w') = 0$ and $d''(h,h') = 0$,
   iff $w = w^{'}$ and $h = h^{'}$, so $(w,h) = (w',h')$.
3. Because, $((w,h),(w',h'))$, $= d'(w,w') + d''(h,h') = d''(h,h') + d'(w,w') = d((w',h'),(w,h))$

Thus, $d((w,h),(w',h')) = d((w',h'),(w,h))$.

$$\begin{aligned} d((w,h),(w',h')) &= d'(w,w') + d''(h,h') \le d'(w,w'') + d''(h,h'') + d'(w'',w') + d''(h'',h') \\ &\le [d'(w,w'') + d''(h',h'')] + [d'(w'',w') + d''(h'',h')] \\ &\le d((w,h),(w'',h'')) + d((w'',h''),(w',h^6)) \end{aligned}$$

Thus, $d((w,h),(w',h')) \le d((w,h),(w'',h'')) + d((w'',h''),(w',h'))$.

It follows that $(W \times H, d)$ is a metric space from 1, 2, 3, and 4.

Corollary 3.4: Suppose $z: W \to W$ be a mapping. About every $k \in \mathbb{N}$, if z has the K-shadowing characteristic. Thus, z^{k+1} has K-shadowing characteristic.

Proof: The same way to prove theorem 3.3.

Theorem 3.5: Assume that (W, d') and (H, d'') are metric spaces with $z: W \to W$ and $\upsilon: H \to H$ are mappings. If z and υ both possess the K-shadowing characteristic, thus $z \times \upsilon$ has the K-shadowing characteristic.

Proof: If (W, d') and (H, d'') are metric spaces with $z: W \to W$ and $\upsilon: H \to H$ are mappings, respectively, we select the metric d on $W \times H$ as follows:

For $\upsilon = (\upsilon_1, \upsilon_2), \mu = (\mu_1, \mu_2) \in W \times H, d(\upsilon, \mu) = d'(\upsilon_1, \mu_1) + d''(\upsilon_2, \mu_2)$.

If z has K-shadowing, for any chosen γ'-pseudo-orbit with $\{w_k\}_{k=0}^{\infty}$ of z with $0 \le \gamma' \le \lambda'$ and any $\eta \in C(W)$ reassuring: $g\gamma' - \inf_{w\in W} d'(\sigma(w), z(w)) \le \lambda'$, then occurs a genuine orbit with $\{\xi_k\}_{k=0}^{\infty}$ of η and

$$d'(w_k, \xi_k) \le \xi'(\gamma' - \inf_{w\in W} d'(\sigma(w), z(w))) \; \forall k \in \mathbb{N}$$

Given that υ has K-shadowing, for chosen $g\gamma''$-pseudo-orbit $\{\mathrm{h}_k\}_{k=0}^{\infty}$ of υ as well as any $0 \le \gamma'' \le \lambda''$ and any $\sigma \in C(W)$ reassuring: $\gamma'' - \inf_{w\in W} d''(\sigma(w), z(w)) \le \lambda''$ it has a genuine orbit of $\{\psi_k\}_{k=0}^{\infty}$ of σ s.t.$d''(h_k, \psi_k) \le \xi''(\gamma'' - \inf_{w\in W} d''(\sigma(h), \upsilon(h)))$ $\forall k \in \mathbb{N}$. So, let's $\xi = \max\{\xi', \xi''\}$,

$\lambda = \min\{\lambda', \lambda''\}$, $\lambda \geq \gamma \geq \gamma' + \gamma'' \geq 0$, as for specific $\gamma\gamma$-pseudo-orbit $\{\upsilon_k\}_{k=0}^{\infty} = \{(\mathrm{w}_k, h_k)\}_{n=0}^{\infty}$ of $z \times v$ and any $\omega \in C(W \times H)$ reassuring:

$$\gamma - \inf_{(w,h)\in W\times H} d(\omega(w,h),(z\times v)(w,h)) \leq \lambda \quad \inf_{(w,h)\in W\times H} d(\omega(w,h),(z\times v)(w,h)) \leq \lambda - \gamma$$

because $\omega(w,h) = (\eta(w,\sigma(h))$

$$\inf_{w\in W} d'(\varnothing(y), g(y)) + \inf_{h\in H} d''(\sigma(h), v(h)) \leq \lambda - \gamma$$

Consequently $\inf_{w\in W} d'(\varnothing(y), g(y)) \leq \lambda - \gamma \leq \lambda' - \gamma'$

Hence a genuine orbit of η with the parameters $\{\xi_k\}_{k=0}^{\infty}$ occurs.

$$d'(\mathrm{w}_k, \xi_k) \leq \xi'(\gamma' - \inf_{w\in W} d''(\sigma(h), v(h)))\ \forall k \in \mathbb{N}$$

as well $\inf_{h\in H} d''(\sigma(h), v(h)) \leq \lambda - \gamma \leq \lambda'' - \gamma''$

So a genuine orbit of σ with the parameters $\{\psi_k\}_{k=0}^{\infty}$, occurs.

$$d''(h_k, \grave{\mathrm{E}}_k) \leq \xi''(\gamma - \inf_{h\in H} d''(\sigma(h), v(h)))\ \forall k \in \mathbb{N}.$$

As a result, here occurs a genuine orbit of ω with the parameters $\{\upsilon'_k\}_{k=0}^{\infty} = \{(\xi_k, \psi_k)\}_{k=0}^{\infty}$, and

$$\begin{aligned} d(\upsilon_k, \upsilon'_k) &\leq d((\mathrm{w}_k, h_k), (\xi_k, \grave{\mathrm{E}}_k)) \leq d'(\mathrm{w}_k, \xi_k) + d''(h_k, \grave{\mathrm{E}}_k) \\ &\leq \xi'(\gamma'' - \inf_{w\in W} d''(\sigma(h), v(h))) + \xi''(\gamma'' - \inf_{h\in H} d''(\sigma h), v(h))) \\ &\leq \xi(\gamma' - \inf_{w\in W} d'(\sigma(h), v(h))) + \xi(\gamma'' - \inf_{h\in H} d''(\sigma h), v(h))) \\ &\leq \xi(\gamma' + \gamma'' - \inf_{w\in W} d'(\sigma(h), v(h)) - \inf_{h\in H} d''(\sigma h), v(h))) \\ &\leq \xi(\gamma - \inf_{(w,h)\in W\times H} d(\omega(\mathrm{w}, h), (z\times v)(\mathrm{w}, h))). \end{aligned}$$

Therefore, $z \times v$ has the K- characteristic.

Remark 3.6: Assume that (W, d') and (H, d'') are metric spaces with $z: W \to W$ and $v: H \to H$ are mappings. If z and v both possess the K-shadowing characteristic, thus $z^2 \times v$ and $z \times v^2$ has the K-shadowing characteristic.

Theorem 3.7: Assume that (W, d') and (H, d'') are metric spaces with $z: W \to W$ and $v: H \to H$ are mappings. If z and v both possess the K-shadowing characteristic, thus $z + \mathrm{v}$ has the K-shadowing characteristic.

Proof: In the same way as proof of the Theorem 3.5.

Remark 3.8: The relationship between bi-shadowing property and K-shadowing property are independent notions, for example let $z: (W, d) \to (W, d)$ be a map and we take $w = \mathbb{R}^+$ where $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ as a subspace of $\mathbb{R}$ then z is K-shadowing property, but not bi-shadowing property because there is not exist a true orbit $\{w_n\}_{n=0}^{\infty}$ of $\mathbb{R}^+$ such that $d(\ell_n, w_n) \leq \xi(\gamma + \sup_{w\in W} d(\sigma(w), z(w)))\forall n \in \mathbb{N}$. Also let $z: (W, d) \to (W, d)$ be a map and we take $w = (0,1)$ then z is bi-shadowing property, but not K-shadowing property because there is not exist γ-pseudoorbit $\{w_k\}_{k=0}^{\infty}$ of σ such that, $\forall k \in \mathbb{N}$, $d(h_k, w_k) \leq \xi(2\gamma - \inf_{w\in W} d(\sigma(w), z(w)))$.

4. Conclusion

A new concept of K-shades was generalized and all the properties related to this concept were studied and we obtained the following. The general reluctance of four types of K-shadowing features is determined in this research that under conditions of composite, addition, power and product. In future work we recommend studying the quasi bi-shadowing and quasi K-shadowing with properties related them.

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