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# Solution of fractional Laplace type equation in conformable sense using fractional fourier series with separation of variables technique

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**Abstract**

The fractional Fourier series method with the separation of variables technique has been applied to solve the fractional Laplace type equation. We use the conformable fractional derivative to study the fractional Laplace type equation and solve it using the conformable fractional Fourier series method with separation of variables and tensor product technique in Banach spaces.

*Keywords:* Fractional Fourier series, conformable fractional derivative, atomic solution

## 1. Introduction

Fractional calculus is concerned with derivatives and integrals of any order [1]. Fractional calculus has been used in practically every discipline of science and engineering over the last four decades. There has been a lot of attention to fractional differential equations in recent years [6, 9]. One of the most significant instruments in applied sciences is the Fourier series. The Fourier series, for example, can be used to solve partial differential equations [5].

The purpose of this study is to apply conformable fractional Fourier series with separation of variable for the fractional Laplace type equation. As an application, we use tensor product technique of Banach spaces to find atomic solution for this equation [7, 8], since there are partial differential

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equations which are not linear so separation of variables does not work. Or there are linear partial differential equations where we can't separate variables, so we need atomic solution in such cases.

The organization of the paper is as follows. In Section 2, the basic concepts of fractional calculus and fractional Fourier series are introduced. In Section 3, we present the fractional Fourier series solution of fractional Laplace type equation. Tensor product technique of Banach spaces are used in Section 4. Finally, Section 5 is devoted to our conclusions.

## 2. Preliminaries

Let us give some needed definitions and theorems that we need in this paper.

**Definition 2.1:** [2] *Given a function* $f:[0,\infty)\to\mathbb{R}$, $t>0$ *and* $\alpha\in(0,1)$. *The conformable derivative of f with respect to t of order α is defined by:*

$$D_t^{\alpha}(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}.$$

If *f* is $\alpha$-differentiable in some $(0,a)$, $a>0$, and $\lim_{t\to 0^+} D_t^{\alpha}(f)(t)$ exists, then $D_t^{\alpha}(f)(0)=\lim_{t\to 0^+} D_t^{\alpha}(f)(t)$.

**Definition 2.2:** [10] *Let X and Y be two Banach spaces and X* be the dual of X. Let* $x\in X$ *and* $y\in Y$. *Then we define an operator* $T:X^*\to Y$ *such that:*

$$T(x^*)=x^*(x)y,\ \forall x^*\in X^*.$$

Moreover, $T$ is a bounded linear operator. We write $x\otimes y$ for $T$, which is called an atom. Atoms are among the main ingredient in the theory of tensor products. Atoms are used in theory of best approximation in Banach spaces.

**Lemma 2.1** [3] *Let* $x_1\otimes y_1$ *and* $x_2\otimes y_2$ *be two non-zero atoms in* $X\otimes Y$ *such that:*

$$x_1\otimes y_1+x_2\otimes y_2=x_3\otimes y_3.$$

Then either $\{x_1,x_2\}$ or $\{y_1,y_2\}$ are linearly dependent.

**Definition 2.3:** [4] *A function f(t) is called α-periodical with period p if:*

$$f(t)=g(\Phi(t))=g\left(\Phi(t)+\frac{p^{\alpha}}{\alpha}\right),\quad \text{for all } t\in 0,\infty).$$

**Definition 2.4:** [4] *Let* $f:[0,\infty)\to\mathbb{R}$ *be a given piecewise continuous function which is α-periodic with period p. Then the α-fractional Fourier series of f associated with the interval* [0, *p*] *is*

$$S(f)(t)=\frac{a_{\circ}}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(n\frac{t^{\alpha}}{\alpha}\right)+b_n\sin\left(n\frac{t^{\alpha}}{\alpha}\right)\right],$$

where

$$a_n=\frac{2\alpha}{p^{\alpha}}\int_0^p f(t)\cos\left(n\frac{t^{\alpha}}{\alpha}\right)\frac{dt}{t^{1-\alpha}},\ n=0,1,2,3,...$$

which is called the cosine $\alpha$-Fourier coefficients of *f*, and

$$b_n=\frac{2\alpha}{p^{\alpha}}\int_0^p f(t)\sin\left(n\frac{t^{\alpha}}{\alpha}\right)\frac{dt}{t^{1-\alpha}},\ n=1,2,3,...$$

the sine α-Fourier coefficients of *f*.

## 3. Fractional Fourier Series Method for Fractional Laplace Type Equation

Our main object in this section is to find a general solution of the fractional Laplace type equation using fractional Fourier series method with separation of variables technique.

Consider the linear fractional Laplace type equation of the form:

$$D_x^{2\alpha}U(x,y)+D_x^{\alpha}D_y^{2\beta}U(x,y)+D_x^{\alpha}D_y^{\beta}U(x,y)=U(x,y),\ 0<\alpha,\beta<1 \tag{1}$$

with value conditions:
$U(x,0)=U(x,1)=0$,
$U(0,y)=0$, $U_x^{\alpha}(1,y)=f(y)$, where $f$ is given.
Let $U(x,y)=P(x)Q(y)$. Substituting in equation (1), to get:

$$P^{2\alpha}(x)Q(y)+P^{\alpha}(x)Q^{2\beta}(y)+P^{\alpha}(x)Q^{\beta}(y)=P(x)Q(y). \tag{2}$$

Simplifying equation (2), we get:

$$[P^{2\alpha}(x)-P(x)]Q(y)=-P^{\alpha}(x)[Q^{2\beta}(y)+Q^{\beta}(y)]. \tag{3}$$

Since $x$ and $y$ are independent variables, then we obtain:

$$\frac{P^{2\alpha}(x)-P(x)}{P^{\alpha}(x)}=\frac{-[Q^{2\beta}(y)+Q^{\beta}(y)]}{Q(y)}=\lambda, \tag{4}$$

where $\lambda$ is a constant to be determined.

Consequently,

$$(*)\begin{cases}Q^{2\beta}(y)+Q^{\beta}(y)+\lambda Q(y)=0,\\ Q(0)=Q(1)=0.\end{cases} \text{ and } (**)\begin{cases}P^{2\alpha}(x)-\lambda P^{\alpha}(x)-P(x)=0,\\ P(0)=0.\end{cases}$$

We start with problem (*). Then the auxiliary equation is: $r^2+r+\lambda=0\Rightarrow\Delta=1-4\lambda$.

Now, there are three possibilities for $\lambda$.

(a) If $\Delta=0$, then $\lambda=\frac{1}{4}$. So, $r_{1,2}=\frac{-1}{2}$. Thus

$$Q(y)=\left(c_1\frac{y^{\beta}}{\beta}+c_2\right)e^{-\frac{y^{\beta}}{2\beta}}. \tag{5}$$

Using initial conditions $Q(0)=Q(1)=0$, we obtain: $Q(y)=0$, which is the trivial solution of (*). Hence, there is no nontrivial solution when $\lambda=\frac{1}{4}$.

(b) If $\Delta>0$, then $r_{1,2}=\frac{-1\pm\sqrt{1-4\lambda}}{2}$. So

$$Q(y)=c_1e^{r_1\frac{y^{\beta}}{\beta}}+c_2e^{r_2\frac{y^{\beta}}{\beta}}, \tag{6}$$

using initial conditions $Q(0)=Q(1)=0$, we obtain that: $Q(y)=0$, which is the trivial solution of (*). Hence, there is no nontrivial solution when $\lambda<\frac{1}{4}$.

(c) If $\Delta < 0$, then $r_{1,2} = \dfrac{-1 \pm i\sqrt{4\lambda - 1}}{2}$. So

$$Q(y) = e^{-\frac{y^\beta}{2\beta}}\left[c_1 \cos\left(\frac{\sqrt{4\lambda-1}}{2}\frac{y^\beta}{\beta}\right) + c_2 \sin\left(\frac{\sqrt{4\lambda-1}}{2}\frac{y^\beta}{\beta}\right)\right], \tag{7}$$

using the initial conditions:

$$Q(0) = 0 \Rightarrow c_1 = 0.$$

$$Q(1) = 0 \Rightarrow \lambda_n = \frac{(2n\pi\beta)^2 + 1}{4}.$$

Thus,

$$Q_n(y) = e^{-\frac{y^\beta}{2\beta}} \sin(n\pi y^\beta). \tag{8}$$

Now, return back to problem (**):

Substitute $\lambda_n$ in problem (**), we get:

$$P^{2\alpha}(x) - \frac{(2n\pi\beta)^2 + 1}{4} P^\alpha(x) - P(x) = 0, \tag{9}$$

the auxiliary equation is: $r^2 - \lambda_n r - 1 = 0 \Rightarrow \Delta = \lambda_n^2 + 4 > 0$. So, $r_{1,2} = \dfrac{\lambda_n \pm \sqrt{\Delta}}{2}$.

Hence, the solution is:

$$P(x) = c_1 e^{r_1 \frac{x^\alpha}{\alpha}} + c_2 e^{r_2 \frac{x^\alpha}{\alpha}}, \tag{10}$$

for some constants $c_1, c_2$.

Using conditions implies that: $P(0) = 0 \Rightarrow c_2 = -c_1$. Thus

$$P(x) = c_1 (e^{r_1 \frac{x^\alpha}{\alpha}} - e^{r_2 \frac{x^\alpha}{\alpha}}). \tag{11}$$

Hence,

$$P_n(x) = c_n (e^{r_1 \frac{x^\alpha}{\alpha}} - e^{r_2 \frac{x^\alpha}{\alpha}}). \tag{12}$$

Now, the general solution of the equation (1), is:

$$U(x,y) = \sum_{n=1}^{\infty} c_n (e^{r_1 \frac{x^\alpha}{\alpha}} - e^{r_2 \frac{x^\alpha}{\alpha}}) e^{-\frac{y^\beta}{2\beta}} \sin(n\pi y^\beta). \tag{13}$$

Using non-homogeneous condition $U_x^\alpha(1,y) = f(y)$, we obtain:

$$f(y) = \sum_{n=1}^{\infty} c_n (r_1 e^{\frac{r_1}{\alpha}} - r_2 e^{\frac{r_2}{\alpha}}) e^{-\frac{y^\beta}{2\beta}} \sin(n\pi y^\beta). \tag{14}$$

Using the $\beta$-fractional Fourier series of $f(y)$, we find that:

$$c_m = \frac{< f(y), e^{-\frac{y^\beta}{2\beta}} \sin(m\pi y^\beta) >}{(r_1 e^{\frac{r_1}{\alpha}} - r_2 e^{\frac{r_2}{\alpha}}) \left\| e^{-\frac{y^\beta}{2\beta}} \sin(m\pi y^\beta) \right\|^2}, \tag{15}$$

where

$$< f(y), e^{-\frac{y^\beta}{2\beta}} \sin(m\pi y^\beta) > = \int f(y) e^{-\frac{y^\beta}{2\beta}} \sin(m\pi y^\beta) d^\beta y, \tag{16}$$

and

$$\left\| e^{-\frac{y^\beta}{2\beta}} \sin(m\pi y^\beta) \right\|^2 = \int e^{-\frac{y^\beta}{\beta}} \sin^2(m\pi y^\beta) d^\beta y. \tag{17}$$

So, we get the general solution of the given problem:

$$U(x,y) = \sum_{m=1}^{\infty} c_m (e^{r_1 \frac{x^\alpha}{\alpha}} - e^{r_2 \frac{x^\alpha}{\alpha}}) e^{-\frac{y^\beta}{2\beta}} \sin(m\pi y^\beta). \tag{18}$$

## 4. Applications

There are partial differential equations which are not linear so separation of variables does not work. Or there are linear partial differential equations where we can't separate variables, so we need atomic solution in such cases. Our main object in this section is to find an atomic solution of the fractional Laplace type equation given in section 3. Hence, we try to find an atomic solution of this equation, which is mean a solution of the form $U(x,y) = P(x)Q(y)$.

Let $U(x,y) = P(x)Q(y)$, substituting in equation (1), to get:

$$P^{2\alpha}(x)Q(y) + P^{\alpha}(x)[Q^{2\beta}(y) + Q^{\beta}(y)] = P(x)Q(y). \tag{19}$$

This can be written in tensor product form as:

$$P^{2\alpha}(x) \otimes Q(y) + P^{\alpha}(x) \otimes [Q^{2\beta}(y) + Q^{\beta}(y)] = P(x) \otimes Q(y). \tag{20}$$

Since we have sum of two atoms is an atom, so we have two cases to consider:

**Case (i):** $P^{2\alpha}(x) = P^{\alpha}(x)$.
**Case (ii):** $Q(y) = Q^{2\beta}(y) + Q^{\beta}(y)$.

**Solution of case (i):**

$$P^{2\alpha}(x) = P^{\alpha}(x) \Rightarrow r^2 - r = 0 \Rightarrow r(r-1) = 0 \Rightarrow \begin{cases} r = 0 \\ or \\ r = 1. \end{cases}$$

So;

$$P(x) = c_1 + c_2 e^{\frac{x^\alpha}{\alpha}}, \tag{21}$$

using initial condition $P(0) = 0 \Rightarrow c_2 = -c_1$. Thus

$$P(x) = c_1(1 - e^{\frac{x^\alpha}{\alpha}}). \tag{22}$$

Now, replace $P(x)$ in equation (18), to get:

$$Q^{2\beta}(y) + Q^\beta(y) = 0. \tag{23}$$

So, the solution is:

$$Q(y) = c_1 + c_2 e^{\frac{-y^\beta}{\beta}}. \tag{24}$$

Using initial condition $Q(0) = 0 \Rightarrow c_2 = -c_1$. Thus

$$Q(y) = c_1(1 - e^{\frac{-y^\beta}{\beta}}). \tag{25}$$

Therefore, we get the atomic solution of (18) as:

$$U(x,y) = P(x)Q(y) = (1 - e^{\frac{x^\alpha}{\alpha}})(1 - e^{\frac{-y^\beta}{\beta}}). \tag{26}$$

**Solution of case (ii):**

$Q^{2\beta}(y) + Q^\beta(y) - Q(y) = 0 \Rightarrow r^2 + r - 1 = 0 \Rightarrow \Delta = 5 > 0$. So, $r_{1,2} = \dfrac{-1 \pm \sqrt{5}}{2}$. Hence,

$$Q(y) = c_1 e^{r_1 \frac{y^\beta}{\beta}} + c_2 e^{r_2 \frac{y^\beta}{\beta}}, \tag{27}$$

using condition $Q(0) = 0$, we get:

$$Q(y) = c_1(e^{r_1 \frac{y^\beta}{\beta}} - e^{r_2 \frac{y^\beta}{\beta}}). \tag{28}$$

Now, for $Q(1) = 0$ gives $c_1 = 0$. Thus

$$Q(y) = 0.$$

So, the second case gives a trivial solution.

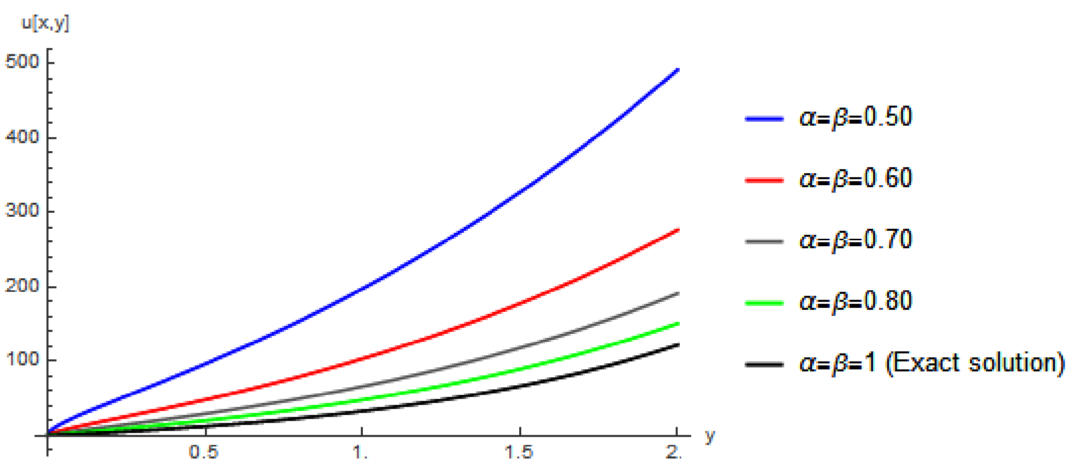


Figure 1: The exact and approximate solutions of $u(x,y)$ for equation (18), when $\alpha = \beta$ with $x = 3$.

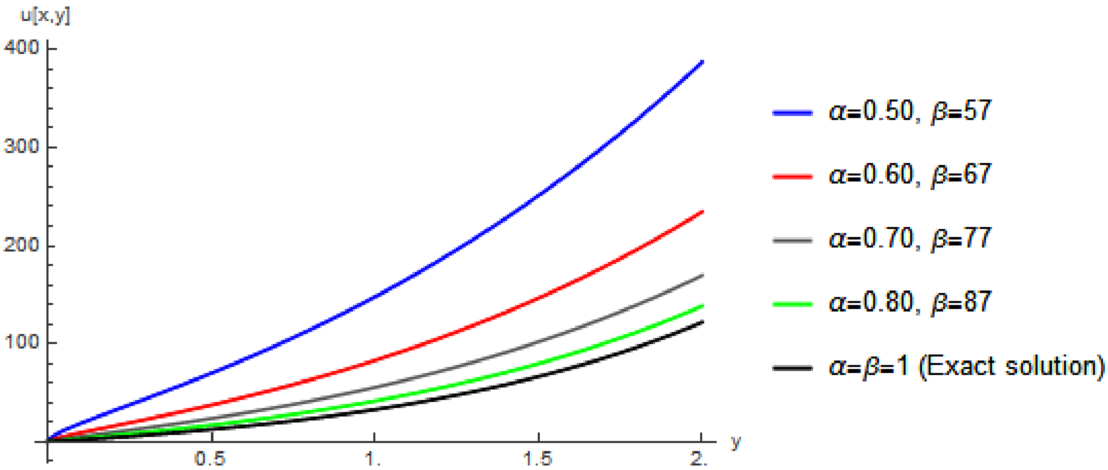


Figure 2: The exact and approximate solutions of $u(x,y)$ for equation (18), when we take different values of fractional order $\alpha$ and $\beta$ with $x=3$.

## 5. Conclusion

The results presented in the preceding sections demonstrate that the separation variables using fractional Fourier series methodology addresses various challenging problems that cannot be solved using traditional methods. We conclude that the conformable fractional Fourier series method is one of very efficient and powerful techniques for finding the solutions of the fractional differential equations.

## References

[1] Abdeljawad, T., *On conformable fractional calculus*. Journal of computational and Applied Mathematics, 279, (2015), 57–66.

[2] Khalil, R., Al-Horani, M., Yousef, A., and Sababheh, M., *A New definition of fractional derivative*. Journal of Computational and Applied Mathematics, 264, (2014), 65–70.

[3] Khalil, R., *Isometries of* $L^{P^*}\hat{\otimes}L^P$. Tamkang Journal of Mathematics, 16 (2), (1985), 77–85.

[4] Abu Hammad, M. A., and Khalil, R., *Fractional Fourier series with applications*. American Journal of Computational and Applied Mathematics, 4 (6), (2014), 187–191.

[5] Cenesiz, Y., and Kurt, A., *The solutions of time and space conformable fractional heat equations with conformable Fourier transform*. Acta Universitatis Sapientiae, Mathematica, 7 (2), (2015), 130–140.

[6] Podlubny, I., Fractional Differential Equations, vol. 198 of Mathematics in Science and Engineering, Academic Press, New York, (1999).

[7] Kadri, I., Horani, M., and Khalil, R., *Tensor product technique and fractional differential equations*. Journal of Semigroup Theory and Applications, 2020, (2020), Article-ID 6.

[8] Khalil, R., and Abdullah, L., *Atomic solution of certain inverse problems*. European Journal of Pure and Applied Mathematics, 3 (4), (2010), 725–729.

[9] Ilhem, K., Al Horani, M., and Khalil, R. R., *Solution of Non-linear Fractional Burger's type Equations using the Laplace Transform Decomposition Method*. Results in Nonlinear Analysis, 5 (2), (2022), 131–151.

[10] Light, W., and Cheney, E., Approximation Theory in Tensor Product Spaces, 1169, New York: Springer, (1985),