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The mixed slicing structure property

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Abstract

The aim of this paper is to structure a new concept of Mixed Slicing Structure Property denoted by (MSSP via short) and we give proof that the mixed fibration is not mixed Hurewics fibration.

Key words and phrases: Mixed fibration, Mixed Slicing structure, G-bundle, Mixed Covering Homotopy Property, Mixed Locally Equi-Connected.

Mathematics Subject Classification (2010): 14Dxx

1. Introduction

The fundamental distinction from modern conceptions of fiber spaces. Seifert believed to be now known as the base space (topological space) of a fiber (topological) space B was a quotient space of B and not a component of the structure. Hassler Whitney first described fiber space in 1935 [1] under the term sphere space. Later, he modified the name to sphere bundle [2]. Jean-Pierre Serre [3], and others are credited with the theory of fibered spaces, which includes fibered manifolds, principal bundles, vector bundles, and topological fibrations as special cases. A fiber structure (E, P, B) is a triple consisting of two topological spaces E, B, and $P : E \to B$ a continuous surjection. The total (or fibered) space refer to space E, P is termed the projection , and B is the base space for each $b_0 \in B$ the set $F = P^{-1}(b_0)$ and F is called fiber over b_0. The fiber structure over B refers to (E, P, B) [4]. A fibration is a continuous mapping satisfying the covering homotopy property, and its CW-complex is called Serre vibration. The slicing structure property (SSP) introduced by [5] if we have given map p: $E \to X$ has the SSP then E is called sliced fiber space over X relative to p and in particular, every bundle space is a sliced fiber space. In addition, the triple $\xi = (E, p, X)$ is called a Hu fiber space (for

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more details see [6]). We investigate that most of the theorems that are valid for fibration (Serre fibration) are also valid for Mixed fibration (Mixed Serre fibration). In particular, with Mixed Slicing Structure Property.

2. Preliminaries

In this section, we illustrate some basic definitions, properties, and results.

Definition 2.1. [7] *Let S and T be topological spaces. If p and q are mapping of S into T, then p and q are homotopic* ($p \simeq q$) *iff there exists a mapping* $h : S \times I^1 \to T$ *such that* $h(e, 0) = p(e)$ *and* $h(e, 1) = q(e)$ *for all* $e \in S$. *The mapping h is called a homotopy between p and q.*

Definition 2.2. [6, 8] *Let* $p : E \to B$ *be a map (fiber bundles), we say that p has Covering Homotopy Property (C.H.P.by short) with respect to X iff given a map* $f : X \to E$ *and* $h_t : X \to B$ *is homotopy such that* $p \circ f = h_0$. *Then there exist a homotopy* $h_t^* : X \to E$ *such that* (1) $h_0^* = f$. (2) $p \circ h_t^* = h_t$, *for all* $x \in X$ *and* $t \in I$. *I is the unit interval.*

Definition 2.3. [9] *Given a fiber space (fiber bundle)* $p : E \to B$, *a section s is a continuous map* $s : B \to E$ *such that pos = identity* $: B \to B$.

Definition 2.4. [6] *The map p is said to be (Hurewicz) Fibration if it has covering homotopy property with respect to all spaces X.*

Definition 2.5. [6, 10] *Let* $p : E \to B$ *be a continuous map of spaces, p has the covering homotopy property (C.H.P) with respect to all CW-complex spaces X is called Serre Fibration.*

Definition 2.6. [11] *(1) Let* E_1, E_2, *X be three topological spaces, let* $Ei = \{E_1, E_2\}$, $f_i = \{f_1, f_2\}$ *where* $f_1 : E_1 \to X$, $f_2 : E_2 \to X$ *are two maps, and* $\alpha : E_2 \to E_1$ *such that* $f_1 \circ \alpha = f_2$ *then* (E_i, f_i, X, α) *is a Mixed fibre space (M-fibre space).*

If $E_1 = E_2 = E$, $\alpha =$ *identity,* $f_1 = f_2 = f$ *then* (E, f, X) *is the usual fibre space.*

(2) Let $\{E_i, f_i, X, \alpha\}$ *be a M-fiber space, let* $x_0 \in X$ *then* $f = \{f_i^{-1}(x_0)]$ *is the M-fibre over* x_0.

Definition 2.7. *Given a Mixed fiber space (M-fibre space)* (E_i, f_i, X, α), *a Mixed section* s_i *is a continuous map* $s_i : X \to E_i$ *such that* $f_i o s_i =$ *identity* $: X \to X$.

Definition 2.8. *Let* $\{E_i, f_i, X, \alpha\}$ *be a M-fibre space, where* $i = 1, 2$. *X, B be a CW-complex spaces and* $h_t : B \to X$ *be map. A continuous* $k_1 : B \to E_1$ *and* $k_2 : B \to E_2$ *such that* $f_1 \circ k_1 = h_t$ *and* $f_2 \circ k_2 = h_t$, *where* $K_i = \{k_1, k_2\}$ *is called a Mixed-covering (M-covering) of* h_t.

Definition 2.9. [1] *Let Y be a CW-complex space,* $f_1 : E_1 \to Y$, $f_2 : E_2 \to Y$, $\alpha : E_2 \to E_1$ *are maps of a spaces such that* $f_1 \circ \alpha = f_2$, *let* $E_i = \{E_1, E_2\}$ *where* $i = 1, 2$. $f_i = \{f_1, f_2\}$ *the quartic* $\{E_i, f_i, Y, \alpha\}$ *has the Mixed covering homotopy property (M-CHP) with respect to a CW-complex X iff given a map* $k : X \to E_2$ *and a homotopy* $ht : X \to Y$ *such that* $f_2 \circ k = h_0$, *then exists a homotopy* $g_t : X \to E_1$ *such that*

(1) $f_1 \circ g_t = h_t$. *(2)* $\alpha \circ k = g_0$.

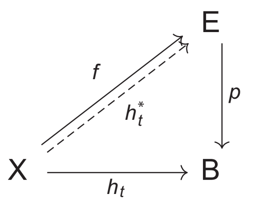


Figure 1. Covering Homotopy Property

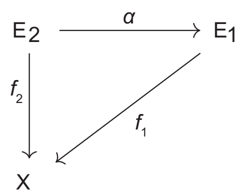


Figure 2. M-fibre space

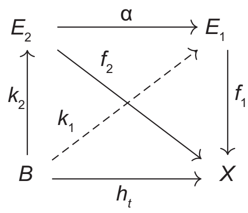


Figure 3. M-Covering

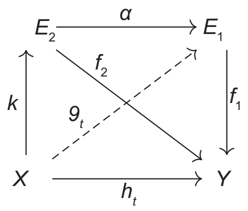


Figure 4. M-Serre Fibration

(1) *M-fiber space is called M-Serre fibration, is it has the (M-CHP) with respect to all CW-complex Y.*
(2) *M-fiber space is called Mixed-Hurewics fibration, if it has mixed covering homotopy property with respect to all spaces.*
(3) *M-fiber space is called Mixed-fibration, if it has mixed covering homotopy property with respect to classes of space.*

3. Non-Trivial of Mixed Principal Bundle

In this section, we give examples for the projection map which is mixed fibration but not a mixed Hurewics fibration.

The projection $p_i : E_i \to B$, where $i = 1, 2$, gives a mixed trivial G-bundle over B, since $E_i|_{U_i} \simeq E_{iU_i}$ and $E_i|_{V_i} \simeq E_{iV_i}$ are mixed trivial G-bundles. To prove that p_i is not a Mixed Hurewics fibration. suppose that B is contractible. Also, if $p_i : E_i \to B$ were mixed fibration, the admit a section it would be necessary. In particular, let $H_t : B \to B$ be a null-homotopy of id_B . Distinctly, we can lift the constant map H_1 to E_i. suppose that p_i is Mixed Hurewics fibration, the mixed covering homotopy property then we get a lift $H_t^*: B \to E_i$, of H to E_i, and as a result a section H_0^* of p_i. But we showed previously that p_i admits no section. Therefore p_i is not a mixed Hurewics fibration. Assume $G = GL_{+1}(\mathbb{R}) = (\mathbb{R}, +,.)$ be the topological abelian group which given by the positive reals with multiplication. Let the trivial G-bundles over U_i and V_i, given by $E_{iU_i} = U_i \times G$ and $E_{iV_i} = V_i \times G$, respectively. The principal G-bundle E over B whose construct by gluing E_{iU_i} and E_{iV_i} along $U_i \cap V_i$ by the G-isomorphism

$$\varphi_{fi} : E_i U_i \,|\, U_i \cap Vi \xrightarrow{\simeq} E_i V_i \,|\, V_i \cap V_i \tag{1}$$

defined by

$$\varphi f_i(x_i, g) = (x_i, f_i(x_i).g) \tag{2}$$

More concretely, Ei is obtained from $E_{iU_i} \amalg E_{iV_i}$ by identifying $(x_i, g) \in E_{iU_i}$ with $\varphi f_i(x_i, g) \in E_{iV_i}$ for each $x \in U_i \cap V_i$ for $x_i > 0$, the continuous function $f : U_i \cap V_i \to \mathbb{R}_+$ which does not extend to a continuous function $B \to \mathbb{R}^+$. Where B be the space obtained by gluing to copies of $\mathbb{R}$ along $\mathbb{R}^+$

$$B = (R \times \{0, 1\})/\mathbb{Z}_2$$

This space is not Hausdorff, let $q : R \times \{0, 1\} \to B$ be the equation map. Define two subset covering B by $U_i = q_i(\mathrm{R} \times \{0\})$ and $V_i = q_i(\mathrm{R} \times \{1\})$. The projection map $p_i : E_i \to B$ gives a principal G-bundle over B, which comes with canonical isomorphism $E_i \,|\, U_i \simeq E_U$ and $E_i|_{V_i} \simeq E_{iV_i}$. We will now show p does not admit a section. Via the construction of E, section of $p_i : E_i \to B$ determine:

(1) a section of $E_{iU_i} = U_i \times G \to U_i$, and therefore a map $s_{U_i} : U_i \to G = \mathbb{R}^+$; similarly, map $s_{V_i} : V_i \to G = \mathbb{R}^+$;
(2) these maps verify $s_{V_i} = f_i(x_i).s_{U_i}(x_i)$ for each $x_i \in U_i \cap V_i$.

In particular, $f_i(x_i) = s_{V_i}(x_i)/s_{U_i}(x_i)$ for all $x_i \in U_i \cap V_i$. Although, this implies that f_i extends to a continuous function $\bar{f}_i : X_i \to \mathbb{R}^+$ given by

$$\bar{f}_i = s_{V_i}(q_i(g(x_i), 1))/s(q_i(g(x_i), 0))$$

which contradicts the known non-extension property of *fi*.

4. Mixed Slicing Structure Property

The slicing structure property introduced by [5] if we given map $p : E \to X$. Via a slicing structure for p, the collection $S = \{\omega, \phi U\}$ of the following entities:
slicing neighborhoods.

(1) A system $\omega = \{U\}$ of open sets of X which covers X, called the slicing neighborhoods.
(2) A system of maps $\{\phi_U \mid U \in \omega\}$ indexed by the slicing neighborhoods, called the slicing functions, where each ϕ_U, is defined on the subspace $U \times p^{-1}(U)$, of the product space $X \times E$, with images in E in such a way that the following two conditions are satisfied:
(SF1)

$$p\phi_U(a, e) = a, \qquad \textit{where } a \in U,\ e \in p^{-1}(U).$$

(SF2)

$$\phi_U(p(e), e) = e \qquad (e \in p^{-1}(U))$$

If a slicing structure $S = \{\omega, \phi_U\}$ for p exists, we say that $p : E \to X$, has the slicing structure property (SSP).

$$BP \Rightarrow SSP \Rightarrow ParaC\,H\,P$$

If $p : E \to X$ has the SSP then E is called sliced fiber space over X. In addition the triple $\xi = (E, p, X)$ is called a Hu fibre space (for more details see [6]).

Definition 4.1. *Let $E_1 \subset X_1$, $E_2 \subset X_2$ and $p_1, q_1 : X_1 \to Y$, $p_2, q_2 : X_2 \to Y$ be continuous function. If $H : X_1 \times I \to Y$ and $G : X_2 \times I \to Y$ are a continuous function such that $H(x, 0) = p_1(x_1)$, $H(x_1, 1) = q_1(x_1)$ and*

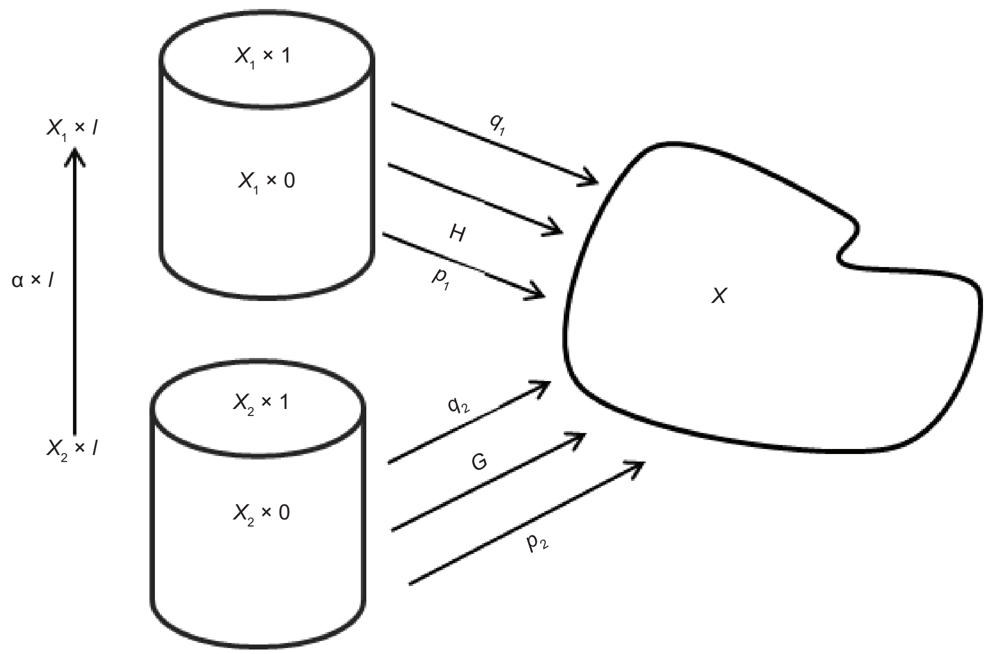


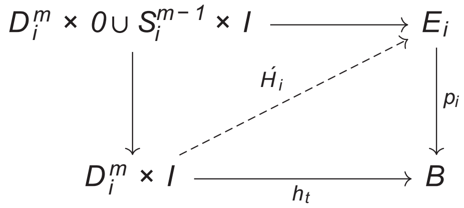
Figure 5. M-relative homotopy

$G(x_2, 0) = p_2(x_2)$, $G(x_2, 1) = q_2(x_2)$ for all $x_1 \in X_1$, $x_2 \in X_2$, and $H(e_1, t) = p_1(e_1) = q_1(e_1)$, $G(e_2, t) = p_2(e_2) = q_2(e_2)$ for all $e_1 \in E_1$, $e_2 \in E_2$ and for all $t \in I$, then H ,G are called M-relative homotopy and p_i is said to be homotopic to q_i relative to E_i, where $i = 1, 2$

Theorem 4.2. *Let $pi : Ei \to B$ be a function where $i = 1, 2$, then the consequently, these:*

(1) *Regarding all m-discs D^m, p_i has the homotopy covering property.*
(2) *Regarding all pair (D^m, S^{m-1}), p_i has M-relative homotopy covering property.*
(3) *p_i has the M-relative homotopy covering property with recpect to all CW-pairs (X, A_i).*

Proof. 1) ⇒ 2) : It is visually obvious, and not hard to prove, that the pair $(D_i^m \times I, D_i^m \times 0 \cup S_i^{m-1} \times I)$ is homeomorphic to the pair $(D_i^m \times I, D_i^m \times 0)$.The desired implication follows easily from this.
2) ⇒ 3) : Suppose that a lift $\acute{H}_i$ is already given on $A_i \times I$. We extend $\acute{H}_i$ over $X_i^m \times I \cup A_i \times I$ by induction on m. At the inductive step, we reduce to constructing a homotopy in a diagram of the form



Such a homotopy exists by assumption (2).
3) ⇒ 1) : This is immediate, taking $Ai = \varnothing$.
Taking B to be a point in (3), we have incidentally proved. This is a proof.

Definition 4.3. *Let $p : E_1 \to X$, $q : E_2 \to X$ and $\alpha : E_2 \to E_1$ be a given maps. By a slicing structure for p, q, we mean a collection $S = \{\omega, \varepsilon, \phi_u, \phi_v\}$ of the following entities:*

(1) *A system $\omega = \{U\}$ and $\varepsilon = \{V\}$ of open sets of X which covers X, called the M-slicing neighborhoods.*
(2) *A system of maps $\{\phi_U \mid U \in \omega\}$ and $\{\phi_V \mid V \in \varepsilon\}$ indexed by the M-slicing neighborhoods, called the M-slicing function, where each ϕ_U, ϕ_V , is defined on the subspace $U \times p^{-1}(U)$, $V \times q^{-1}(V)$ respective of the product space $X \times E_1$, $X \times E_2$ respective with images in E_1, E_2 in such a way that the following two conditions are satisfied:*
(SF1)

$$p\phi_U(a, e_1) = a, \qquad \text{where } a \in U,\ e_1 \in p^{-1}(U)$$

And

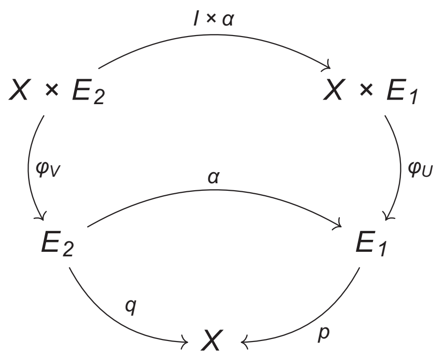
$$q\phi_V(b, e_2) = b, \qquad \text{where } b \in V,\ e_2 \in q^{-1}(V).$$

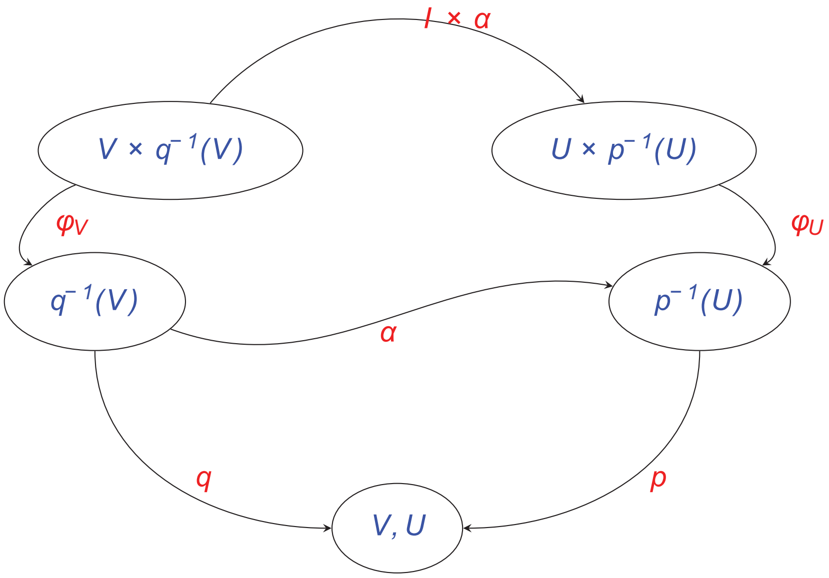
(SF2)

$$\phi_U(p(e_1), e_1) = e_1 \qquad (e_1 \in p^{-1}(U))$$

And

$$\phi_V(q(e_2), e_2) = e_2 \qquad (e_2 \in q^{-1}(V))$$





Definition 4.4. *If a M-slicing structure* $S = \{\omega, \varepsilon, \phi_U, \phi_V\}$ *for p, q exists, we say that* $p : E_1 \to X$, $q : E_2 \to X$ *has the M-slicing structure property (MSSP). If* $p : E_1 \to X$, $q : E_2 \to X$ *has the MSSP then* $\xi = (E_i, p, q, X)$ *is called a Hu M-fibre space, where* $i = 1, 2$.

Definition 4.5. *A space X is Mixed Locally Equi-Connected (M-LEC) if it has a cover by open sets V,U and for each V and U, a connecting maps* $\sigma_1 : U \times U \to X^I$ *and* $\sigma_2 : V \times V \to X^I$ *such that:*
$\sigma_1(x_1, y_1)(0) = x_1$, $\sigma_1(x_1, y_1)(1) = y_1$, *and* $\sigma_1(x_1, x_1)(t) = x_1$.
$\sigma_2(x_2, y_2)(0) = x_2$, $\sigma_2(x_2, y_2)(1) = y_2$, *and* $\sigma_2(x_2, x_2)(t) = x_2$.

Theorem 4.6. *If B is M-Locally Equi-Connected (M-LEC), then each regular M-Serre fibre space* (E_i, p, q, B) *is a Hu M-fibre space.*

Proof. Since (E_i, p, q, B) is a regular M-Serre fibration, we have a regular lifting functions

$$\lambda : \Omega_p \to E_1^I$$

and

$$\delta : \Omega_q \to E_2^I$$

where

$$\Omega_p = \{(e_1, \omega) \in E_1 \times B^I \mid p(e_1) = \omega(0)\},$$

$$\Omega_q = \{(e_2, \varepsilon) \in E_2 \times B^I \mid q(e_2) = \varepsilon(0)\},$$

and

$$\lambda(e_1, \omega)(0) = e_1,\ p\lambda(e_1, \omega) = \omega,$$

$$\delta(e_2, \varepsilon)(0) = e_2,\ q\delta(e_2, \varepsilon) = \varepsilon$$

and if ω, ε are constants, then $\lambda(e_1, \omega)$ and $\delta(e_2, \varepsilon)$ are also a constant. Then we may define

$$\phi_\mu : U_\mu \times p^{-1}(U_\mu) \to E_1,$$

$$\phi_v : V_v \times q^{-1}(V_v) \to E_2.$$

By

$$Q_\mu(a, e_1) = \lambda(e_1, \sigma_\mu(p(e_1), a)(1)),$$
$$Q_v\,(b, e_2) = \delta(e_2, \sigma_v\,(q(e_2), b)(1)).$$

Where $\{U\mu\}$ and $\{Vv\}$ are cover for B and $\{\sigma_\mu\}$, $\{\sigma_v\}$ are connecting maps.
Q_μ, Q_v are a M-slicing function so (E_i, p, q, B) is a Hu M-fibre space.

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