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$F(a_0, a_1, \ldots., a_n)$–structures on manifolds

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Abstract

The aim of the present paper is to study the geometry of n-dimensional differentiable manifolds endowed with the $F(a_0,a_1,\ldots.,a_n)$-structure satisfying $a_nF^n + a_{n-1}F^{n-1} + \ldots\ldots + a_1F + a_0I = 0$ and establish its existence. Also, it is proved that for the complex numbers the dimension of a manifold $\mathcal{M}$ endowed with the $F(a_0,a_1,\ldots.,a_n)$-structure is even. Furthermore, we study the Nijenhuis tensor of a tensor field F of type (1,1) satisfying the general quadratic equation, which is a particular case of the $F(a_0,a_1,\ldots.,a_n)$-structure. At last, we study the integrability conditions of the $F(a_0,a_1,\ldots.,a_n)$-structure.

Key words: Manifold, Distribution, Integrability, Differential equations.

1. Introduction

In [1], Yano developed the idea of an f-structure on a differentiable manifold $\mathcal{M}$ as a tensor field $f\,(\neq 0)$ of type (1, 1) fulfilling $f^3 + f = 0$. Goldberg and Yano [2] proposed the idea of an n-degree polynomial structure of on $\mathcal{M}$ which is generalization the f-structure and investigate its geometric properties. Later on, Debnath and Konar [3] contributed some significant findings by introducing a novel kind of almost quadratic ϕ-structure on a differentiable manifold $\mathcal{M}$. Recently, Gök et al. [4] developed and explored the concepts of a $f_{(a,b)}(3,2,1)$-structure on $\mathcal{M}$. f-structures have also been investigated in [5–14]. Motivated by above mentioned studies, we study the geometry of n-dimensional differentiable manifolds endowed with $F(a_0,a_1,\ldots.,a_n)$-structure and investigate some geometric properties of it.

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2. $\mathcal{F}(a_0,a_1,....,a_n)$-structure

Let $\mathcal{M}$ be an n-dimensional differentiable manifold of the class C^∞ and $\mathcal{F}$ be a non-null tensor field of type (1, 1) satisfying

$$a_n\mathcal{F}^n+a_{n-1}\mathcal{F}^{n-1}+.......+a_1\mathcal{F}+a_0I=0, \tag{2.1}$$

where $(a_0,a_1,....,a_n)$ are non-zero real numbers.

Operating U in (2.1), we infer

$$a_n\mathcal{F}^nU+a_{n-1}\mathcal{F}^{n-1}U+.......+a_1\mathcal{F}U+a_0U=0, \tag{2.2}$$

where $U\in\Im_0^1(\mathcal{M})$.

A manifold $\mathcal{M}$ endowed with $\mathcal{F}(a_0,a_1,....,a_n)$-structure is called $\mathcal{F}(a_0,a_1,....,a_n)$-structure manifold.

Remark: $\Im_0^1(\mathcal{M})$ and $\Im_1^1(\mathcal{M})$ denote a vector field and a tensor field of type (1,1) in $\mathcal{M}$, respectively.

Theorem 2.1. *The rank of $\mathcal{F}\in\Im_1^1(\mathcal{M})$ in an $\mathcal{F}(a_0,a_1,....,a_n)$-structure is equal to the dimension of $\mathcal{M}$.*

Proof. Let $\mathcal{F}U=0\Rightarrow\mathcal{F}^2U=0,\ ....,\ \mathcal{F}^nU=0$.

From (2.2), we acquire

$$a_0U=0\Rightarrow U=0,\ a_0\neq 0.$$

Let $\mathcal{TM}$ be the tangent space of $\mathcal{M}$ and the Kernel of $\mathcal{F}$ is the trivial subspace {0} of $\mathcal{TM}$. The nullity of $\mathcal{F}$ is denoted by υ, then $\upsilon = 0$. If the rank of $\mathcal{F}$ is ρ, then from [15]

$$\upsilon+\rho=n, \tag{2.3}$$

which by $\upsilon = 0$ gives $\rho = n$. This completes the proof.

Theorem 2.2. *For the complex number, the dimension of $\mathcal{M}$ endowed with $\mathcal{F}(a_0,a_1,....,a_n)$-structure is even.*

Proof. Assume that ζ be the eigen value of $\mathcal{F}$ and E be the associated eigen vector.

Then

$$\mathcal{F}E=\zeta E,\ \mathcal{F}^2E=\zeta^2E,........,\mathcal{F}^nE=\zeta^nE.$$

From (2.2), we have

$$a_n\zeta^n+a_{n-1}\zeta^{n-1}+.......+a_1\zeta+a_0I=0,$$

and

$$a_n\zeta^nE+a_{n-1}\zeta^{n-1}\mathrm{E}+.......+a_1\zeta E+a_0E=0. \tag{2.4}$$

The roots of (2.4) will be real and complex numbers and defined by

$$(i)\sum_{i=1}^{n}\zeta_i=-\frac{a_{n-1}}{a_n},$$

$$(ii)\sum_{i,j=1}^{n}\zeta_i\zeta_j=\frac{a_{n-2}}{a_n}, \tag{2.5}$$

$$(iii)\sum_{i_1,i_2,....,i_k}^{n}\zeta_{i_1}\zeta_{i_2}.....\zeta_{i_k}=(-1)^k\frac{a_{n-k}}{a_n}.$$

The complex eigen values of $\mathcal{F}$ are of the form $\alpha\pm\beta i$ and occur in pairs. In the light of this fact, the dim $(\mathcal{M}) = n$ must be even.

Theorem 2.3. *The $\mathcal{F}(a_0,a_1,....,a_n)$-structure on $\mathcal{M}$ is not unique.*

Proof. $\forall\ \mathcal{F}'\in \Im_1^1(\mathcal{M})$ and $U\in \Im_0^1(\mathcal{M})$, we have

$$\mu(\mathcal{F}'(U)) = \mathcal{F}(\mu(U)), \tag{2.6}$$

where $\mu(\neq 0)$ is a vector valued function. We also have

$$\begin{aligned}\mu(\mathcal{F}'^2(U)) &= \mu(\mathcal{F}'(\mathcal{F}'U)),\\ &= \mathcal{F}(\mu((\mathcal{F}'U))),\\ &= \mathcal{F}(\mathcal{F}(\mu(U))),\\ &= \mathcal{F}^2(\mu(U)).\end{aligned}$$

In a similar manner, we infer

$$\mu(\mathcal{F}'^3(U)) = \mathcal{F}^3(\mu(U)),\ ......,\ \mu(\mathcal{F}'^n(U)) = \mathcal{F}^n(\mu(U)).$$

Further, we acquire

$$\begin{aligned}&\mu\{a_n\mathcal{F}'^nU + a_{n-1}\mathcal{F}'^{n-1}U + ... + a_1\mathcal{F}'U + a_0U\}\\ &= a_n\mathcal{F}^n(\mu(U)) + a_{n-1}\mathcal{F}^{n-1}(\mu(U)) + ... + a_1\mathcal{F}(\mu(U)) + a_0\mu(U))\\ &= 0.\end{aligned} \tag{2.7}$$

This implies that

$$a_n\mathcal{F}'^n U + a_{n-1}\mathcal{F}'^{n-1} U + ... + a_1\mathcal{F}'U + a_0U = 0.$$

Thus, $\mathcal{F}'$ gives the $\mathcal{F}'(a_0,a_1,....,a_n)$-structure on $\mathcal{M}$.

3. Existence Conditions

The necessary and sufficient condition for the even dimensional manifold M^{2km} endowed with an $\mathcal{F}'(a_0,a_1,....,a_n)$-structure for only complex numbers is established.

Theorem 3.1 *It is necessary and sufficient that an even dimensional manifold $\mathcal{M}^{2km}$ contains k distributions π_m^j and $\tilde{\pi}_m^j$, j = 1, 2, .., k($\tilde{\pi}_m^j$ is the conjugate of π_m^j) of dimension m in such a way that both are disjoint and span an even dimensional manifold of dimension 2km in order to admit the $\mathcal{F}(a_0,a_1,....,a_n)$-structure for only the complex numbers.*

Proof. Let us consider an even dimensional manifold $\mathcal{M}^{2km}$ endowed with the $\mathcal{F}(a_0,a_1,....,a_n)$-structure for only complex numbers. Then $\mathcal{F}$ contains k sets of m eigen values of the type $\alpha_j + i\beta_j$ and k sets of m eigen values of the type $\alpha_j - i\beta_j$, $j = 1,2,...,k \in N$, where N is a set of natural numbers.

Let P_u^j and Q_u^j, $u = 1,2,...,m$ be m eigen vectors associated to m eigen values $\alpha_j + i\beta_j$ and $\alpha_j - i\beta_j$, respectively of $\mathcal{F}$.

Let

$$b_j^u P_u^j + c_j^u Q_u^j = 0,\ b_j^u, c_j^u \in R. \tag{3.1}$$

Operating $\mathcal{F}$ on (3.1) and using the property that P_u^j, Q_u^j are the eigen vectors associated with $\alpha_j + i\beta_j$ and $\alpha_j - i\beta_j$ of $\mathcal{F}$, we get

$$(b_l^u P_u^l - c_l^u Q_u^l)(b_j^u P_u^j + c_j^u Q_u^j) = 0,\ b_l^u, c_l^u \in R,\ u = 1,2,...,m; l = 1,2,...,k, \tag{3.2}$$

where R is a set of real numbers.

From (3.1) and (3.2), we infer $b_l^u = 0$ and $c_l^u = 0,\ u = 1,2,...,m;\ j = l.$ Thus, the set $\{P_u^l, Q_u^l\}$ is linearly independent. Similarly we find, $b_j^u = 0$ and $c_j^u = 0, \forall u = 1,2,...,m; j = 1,2,...,k.$ Hence the set $\{P_u^l, Q_u^l\}$ is linearly independent.

Let projections L_j and M_j be the linear transformations defined by [16]

$$L_j(U) = F U - (\alpha_j - i\beta_j)U, \tag{3.3}$$

and

$$M_j(U) = F U - (\alpha_j - i\beta_j)U. \tag{3.4}$$

Obiously, we have

$$\begin{aligned} L_j(P_u^j) &= 2i\beta_j P_u^j, \\ L_j(Q_u^j) &= 0, \\ M_j(P_u^j) &= 0, \\ M_j(Q_u^j) &= -2i\beta_j Q_u^j. \end{aligned} \tag{3.5}$$

Thus there are k distributions π_m^j and $\tilde{\pi}_m^j$ with dimension m, all of which are disjoint and span a manifold of dimension $2km$.

On contrary, suppose that there are k distributions π_m^j and $\tilde{\pi}_m^j$ with dimension m, all of which without having common direction and span a manifold of dimension $2km$.

Let us consider P_u^j and Q_u^j be the eigen vectors for k distributions π_m^j and $\tilde{\pi}_m^j$, respectively. Then the set $\{P_u^j, Q_u^j\}$ is linearly independent.

Let $\{p_u^j, q_u^j\}$ be 1-forms dual to $\{P_u^j, Q_u^j\}$ such that

$$\begin{aligned} p_u^j\left(P_v^j\right) &= \zeta_v^u, \\ p_u^j\left(Q_v^j\right) &= 0, \\ q_u^j\left(P_v^j\right) &= 0, \\ q_u^j\left(Q_v^j\right) &= \zeta_v^u, \end{aligned} \tag{3.6}$$

and

$$p_j^u(u)P_u^j + q_j^u(u)Q_u^j = U. \tag{3.7}$$

Operating F on (3.7) and using the fact that P_u^l and Q_u^l are eigen vectors for the eigen values $\alpha_l + i\beta_l$ and $\alpha_l - i\beta_l$ respectively, then we have

$$[(\alpha_l + i\beta_l)p_l^u(U)P_l^l + (\alpha_l - i\beta_l)q_l^u(U)Q_u^l][p_j^u(U)P_u^j + q_j^u(U)Q_u^j] = F U. \tag{3.8}$$

Thus from (3.7) and (3.8), we get

$$F U = \alpha_l U + [i\beta_l(p_l^u(U)P_u^l - q_l^u(U)Q_u^l][p_j^u(U)P_u^j + q_j^u(U)Q_u^j]. \tag{3.9}$$

Now again operating F on (3.8) and using the same fact that P_u^l, Q_u^l are eigen vectors for the eigen values $\alpha_l + i\beta_l$ and $\alpha_l - i\beta_l$ of F, we get

$$F^2 U = [(\alpha_l + i\beta_l)^2(p_l^u(U)P_u^l + (\alpha_l + i\beta_l)^2 q_l^u(U)Q_u^l)][p_j^u(U)P_u^j + q_j^u(U)Q_u^j]. \tag{3.10}$$

Making use of (3.7), (3.9) and (3.10), we infer

$$F^2 U - 2\alpha_l F U + (\alpha_l^2 + i\beta_l^2)U = 0. \tag{3.11}$$

Similarly, we can find F^3, F^4, ...,F^n by considering only the complex roots.

Thus only for the complex numbers, the manifold M^{2km} (even dimension) is endowed with $F(a_0, a_1, ..., a_n)$-structure.

4. The Nijenhuis Tensor

In this section, we study the Nijenhuis tensor of a tensor field F of type (1,1) satisfying the general quadratic equation which is a particular case of the $F(a_0,a_1,....,a_n)$-structure.

Let $F(\neq 0)\in \Im_1^1(\mathcal{M})$ satisfying the general quadratic structure [17]

$$F^2+a_1F+a_0I=0, \tag{4.1}$$

where a_0, a_1 are real numbers.

The Nijenhuis tensor of F is defined by

$$N(U,V)=[FU,FV]+F^2[U,V]-F[FU,V]-F[U,FV], \tag{4.2}$$

$\forall U,V\in \Im_0^1(\mathcal{M})$ [18].

Theorem 4.1. *Let* $F(\neq 0) \in \Im_1^1(\mathcal{M})$ *and* U, $V\in \Im_0^1(\mathcal{M})$. *Then*

$$N(U,FV) = N(FU,V), \tag{4.3}$$

$$N(FU, FV) = a_1^2[FU,FV]+a_0a_1[FU,V]+a_0^2[U,V]-a_0[FU,FV]+a_1F[FU,FV]$$
$$+a_0F[U, FV]+a_0F[FU,V]. \tag{4.4}$$

Proof. By replacing U by FU in (4.2), we have

$$N(FU, V) = [F^2U,FV] + F^2[FU, V] - F[F^2U, V] - F[FU,FV],$$

which by using (4.1) becomes

$$N(FU, V) = -a_1[FU,FV] - a_0[U,FV] - a_0[FU, V] + a_0F[U, V] - F[FU,FV]. \tag{4.5}$$

By replacing V by FV in (4.2) and using (4.1), we have

$$N(U,FV) = -a_1[FU,FV]-a_0[U,FV]-a_0[FU, V]+a_0F[U, V]-F[FU,FV]. \tag{4.6}$$

By replacing U by $FU$and V by FV in (4.2) and using (4.1), we lead to

$$N(FU,FV) = a_1^2[FU,FV] + a_0a_1[FU, V] + a_0^2[U, V] - a_0[FU,FV]$$
$$+ a_1F[FU,FV] + a_0F[U,FV] + a_0F[FU, V]. \tag{4.7}$$

Equations (4.5), (4.6) and (4.7) lead to the proof of the theorem.

5. Integrability Conditions

In this section, we discuss the integrability conditions of $F(a_0,a_1,....,a_n)$-structure with the distributions π_m^j and $\tilde{\pi}_m^j$.

Theorem 5.1. *The necessary and sufficient condition for the k distribution* π_m^j *to be integrable*

$$(dM_j)(U, V) = 0. \tag{5.1}$$

Proof. Suppose the distribution π_m^j is integrable, then [19]

$$U, V\in \pi_m^j \Rightarrow [U, V] \in \pi_m^j .$$

Therefore, we have

$$M_j(U) = 0, M_j(V) = 0, M_j([U, V]) = 0. \tag{5.2}$$

As we know

$$(dM_j)(U, V) = U.M_j(U) - V.M_j(V) -M_j([U, V]), \tag{5.3}$$

which by making the use of (5.2) reduces to

$$(dM_j)(U, V) = 0. \tag{5.4}$$

Thus the condition is necessary.

On contrary, assume that the differential equations $(dM_j)(U, V) = 0$, $\forall U, V \in k$ distributions π_m^j that is the differential equations

$$(dM_j)(U, V) = 0, \forall j = 1, 2, ..., k.$$

Thus, $M_j([U, V]) = 0$, as $M_j(U) = 0 = M_j(V)$.

Now, we have

$$\begin{aligned} Lj([U, V]) &= F[U, V] - (\alpha_j - i\beta_j)[U, V] \\ &= (\alpha_j + i\beta_j)[U, V] - (\alpha_j - i\beta_j)[U, V] \\ &= 2i\beta_j\,[U, V]. \end{aligned}$$

If U,V belong to the distributions π_m^j, then $[U,V]$ also belong to the distributions π_m^j. Thus π_m^j is integrable [20].

Theorem 5.2. *The necessary and sufficient condition for the k distributions* π_m^j *is integrable if and only if differential equations* $(dL_j)(U, V) = 0$, $\forall j = 1, 2, ..., k$.

Proof. Similar to Theorem 5.1, the proof is straightforward.

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