Results in Nonlinear Analysis 6 (2023) No. 1, 107–115
https://doi.org/10.31838/rna/2023.06.01.010
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Some geometric properties on Lorentzian Sasakian manifolds

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Abstract

The objective of the present paper is to study and investigate the geometric properties of Concircular curvature tensor on a Lorentzian Sasakian manifold (in short LS-manifold) endowed with the quarter-symmetric non metric connection. This research is also supported with an example that satisfies the conditions of $\vartheta-$Concircularly flat and Υ-Concircularly flat Lorentzian Sasakian manifold endowed with the quarter-symmetric non metric connection.

Keywords: Lorentzian Sasakian manifolds, Quarter-symmetric metric connection, Concircular curvature tensor, $\Lambda-$Einstein manifold

Mathematics Subject Classification (2010): 53C15, 53C25

1. Introduction

In 1924, Friedmann and Schouten [10] introduced the idea of semi-symmetric connection on a differentiable manifold. Later Matsumoto [14] and Sato [19], introduced the notion of Lorentzian Sasakian manifolds and an almost paracontact manifold respectively. Mihai [15] introduced the same notion independently and obtained several results. The Lorentzian Sasakian manifolds has also been studied in detail by Sato [19], Matsumoto, Mihai [15], De & Shaikh [7] and many others in [3], [17]

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Received February 12, 2023; Accepted March 2, 2023; Online April 12, 2023

and [18]. Some interesting results were obtained for conformally recurrent and conformally symmetric P-Sasakian manifold in [1].

A linear connection $\bar{\nabla}$ on a differentiable manifold M is said to be a semi-symmetric connection if the torsion tensor T of the connection satisfies

$$T(X,Y) = \Lambda(Y)X - \Lambda(X)Y,$$

where Λ is a 1-form and ϑ is a vector field defined by $\Lambda(X) = g(X,\vartheta)$, for all vector fields X on $\Gamma(TM)$. $\Gamma(TM)$ is the set of all differentiable vector fields on M. Semi-symmetric metric and non metric connection on para-Sasakian manifold was studied by [4] and [5]. In 1975, Golab [11] defined and studied quarter-symmetric connection in differentiable manifolds with affine connections. In 2020, Khan [13] studied properties of tangent bundle endowed with quarter-symmetric non-metric connection on an almost Hermitian manifold. Later in 2023, Khan et al. [12] studied properties on lifts of a quarter-symmetric metric connection from a Sasakian manifold to its tangent bundle.

A linear connection $\bar{\nabla}$ on an n-dimensional Riemannian manifold (M, g) is called a quarter-symmetric connection [11], if its torsion tensor T satisfies

$$T(X,Y) = \Lambda(Y)\Upsilon X - \Lambda(X)\Upsilon Y,$$

where Υ is a (1,1) tensor field.

In particular, if $\Upsilon X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection [10].

Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection. If moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) \neq 0, \tag{1}$$

for all X,Y,Z on $\Gamma(TM)$, then $\bar{\nabla}$ is said to be a quarter-symmetric non metric connection (in short qsnmc).

A relation between the quarter-symmetric non metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ in an n-dimensional Lorentzian Sasakian manifold M is given by [8]

$$\bar{\nabla}_X Y = \nabla_X Y - \Lambda(X)\Upsilon Y. \tag{2}$$

The 1-form Λ is defined by $\Lambda(X) = g(X,\vartheta)$ and ϑ is the corresponding vector field. Bagewadi and Venkatesha [2] studied Concircular Υ-recurrent Lorentzian Sasakian manifolds which generalized the notion of locally Concircular Υ-symmetric Lorentzian Sasakian manifolds and obtained some interesting results.

Let $\bar{R}$ and R be the curvature tensors with respect to the quarter-symmetric non metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ respectively. Then, we have

$$\bar{R}(X,Y)Z = R(X,Y)Z + (\Lambda(X)Y - \Lambda(Y)X)\Lambda(Z) + g(Y,Z)\Lambda(X)\vartheta - g(X,Z)\Lambda(Y)\vartheta, \tag{3}$$

$$\bar{R}(\vartheta,Y)Z = -\bar{R}(Y,\vartheta)Z = -2\Lambda(Z)Y - 2\Lambda(Y)\Lambda(Z)\vartheta, \tag{4}$$

$$\bar{S}(Y,Z) = S(Y,Z) - g(Y,Z) - n\Lambda(Y)\Lambda(Z), \tag{5}$$

$$\bar{S}(Y,\vartheta) = 2(n-1)\Lambda(Y), \bar{S}(\vartheta,\vartheta) = -2(n-1), \tag{6}$$

$$\bar{S}(\Upsilon Y,\Upsilon Z) = \bar{S}(Y,Z) - g(Y,Z) - (n-2)\Lambda(Y)\Lambda(Z), \tag{7}$$

for all vector fields $X,Y,Z \in \Gamma(TM)$, where $\bar{S}$ and S be the Ricci tensors with respect to the quarter-symmetric non metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ respectively.

A Riemannian manifold M is locally symmetric if its curvature tensor R satisfies $\nabla R = 0$. As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and studied their generalizations. A Riemannian manifold M is said to be semi-symmetric if its curvature tensor R satisfies $R(X,Y).R = 0$, where $R(X, Y)$ acts on R as a derivation.

A Riemannian manifold M is said to be Ricci-semi symmetric manifold if it satisfies the relation $\bar{R}(X,Y)\,\bar{S} = 0$, where $\bar{R}(X,Y)$ the curvature operator.

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a Concircular transformation. A Concircular transformation is always a conformal transformation. We know, geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of Concircular transformations, i.e., the Concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see also [6]). An interesting invariant of a transformation is the Concircular curvature tensor $\bar{C}$. Some useful properties are given below:

$$\bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{\bar{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{8}$$

Using (8), we obtain

$$\begin{aligned} \tilde{\bar{C}}(X,Y,Z,U) &= \tilde{\bar{R}}(X,Y,Z,U) - \frac{\bar{r}}{n(n-1)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)], \\ \text{and } \tilde{\bar{C}}(X,Y,Z,U) &= g(\bar{C}(X,Y)Z,U),\ \tilde{\bar{R}}(X,Y,Z,U) = g(\bar{R}(X,Y)Z,U), \end{aligned} \tag{9}$$

where $X,Y,Z,U \in \Gamma(TM)$ and $\bar{C}$ is the Concircular curvature tensor and $\bar{r}$ is the scalar curvature with respect to the quarter-symmetric non metric connection respectively. Riemannian manifolds with vanishing Concircular curvature tensor are of constant curvature. Thus the Concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

Putting $X = \vartheta$ in (8) and using (4), we obtain

$$\bar{C}(\vartheta,Y)Z = \left(-\frac{\bar{r}}{n(n-1)}\right)g(Y,Z)\vartheta - \left(\frac{\bar{r}}{n(n-1)} + 2\right)\Lambda(Z)Y - 2\Lambda(Y)\Lambda(Z)\vartheta. \tag{10}$$

2. Preliminaries

An n-dimensional differentiable manifold M is said to be an almost para-contact manifold, if it admits an almost para-contact structure $(\Upsilon,\vartheta,\Lambda,g)$ consisting of a (1, 1) tensor field Υ, vector field ϑ, 1-form Λ and Lorentzian metric g satisfying

$$\Upsilon \circ \vartheta = 0, \quad \Lambda \circ \Upsilon = 0, \quad \Lambda(\vartheta) = -1, \quad g(X,\vartheta) = -\Lambda(X), \tag{11}$$

$$\Upsilon^2 X = -X + \Lambda(X)\vartheta, \tag{12}$$

$$g(\Upsilon X, \Upsilon Y) = g(X,Y) + \Lambda(X)\Lambda(Y), \tag{13}$$

$$(\nabla_X \Lambda)Y = -g(X, \Upsilon Y) = (\nabla_Y \Lambda)X, \tag{14}$$

for any vector field X, Y on M. Such a manifold is termed as Lorentzian para-contact manifold and the structure $(\Upsilon,\vartheta,\Lambda,g)$ a Lorentzian para-contact structure [14].

If moreover $(\Upsilon,\vartheta,\Lambda,g)$ satisfies the conditions

$$d\Lambda = 0,\ \nabla_X \vartheta = -\Upsilon X, \tag{15}$$

$$(\nabla_X \Upsilon)Y = -g(X,Y)\vartheta - \Lambda(Y)X, \tag{16}$$

for X, Y tangent to M, then M is called a Lorentzian Sasakian manifold or briefly LS-manifold, where ∇ denotes the covariant differentiation with respect to Lorentzian metric g.

For the curvature tensor R, the Ricci tensor S and the Ricci operator Q in a LS-manifold M with respect to the Levi-Civita connection the following relation hold

$$\Lambda(R(X,Y)Z) = g(Y,Z)\Lambda(X) - g(X,Z)\Lambda(Y), \tag{17}$$

$$R(\vartheta,X)Y = -g(X,Y)\vartheta - \Lambda(Y)X, \tag{18}$$

$$R(\vartheta,X)\vartheta = -R(X,\vartheta)\vartheta = -X + \Lambda(X)\vartheta, \tag{19}$$

$$R(X,Y)\vartheta = \Lambda(Y)X - \Lambda(X)Y, \tag{20}$$

$$S(X,\vartheta) = (n-1)\Lambda(X), Q\vartheta = -(n-1)\vartheta \tag{21}$$

$$S(\Upsilon X, \Upsilon Y) = S(X,Y) - (n-1)\Lambda(X)\Lambda(Y), \tag{22}$$

for all vector fields $X,Y \in \Gamma(TM)$.

3. Main Results

In this paper, we study a type of quarter symmetric non-metric connection (qsnmc) on LS-manifolds. The paper is organized as:

1. First section includes introduction.
2. Section two is equipped with some prerequisites of a LS-manifold.
3. In section three main results of the paper are discussed.
4. In section four we studied ϑ-Concircularly flat LS-manifold with respect to the qsnmc.
5. Υ-Concircularly flat LS-manifolds with respect to the qsnmc have been studied in section five.
6. In next section, we investigate Ricci-semi symmetric manifolds with respect to the qsnmc of a LS-manifold.
7. At last, we construct an example of a 5-dimensional LS-manifolds endowed with the qsnmc which verify the results of section four and five.

4. ϑ-Concircularly Flat Lorentzian Para-Sasakian Manifold with Respect to the Quarter-Symmetric Non Metric Connection

Definition 4.1. *A LS-manifold is said to be ϑ – Concircularly flat* [2] *with respect to the qsnmc if*

$$\bar{C}(X,Y)\vartheta = 0,$$

where X, $Y \in \chi(M)$.

Theorem 4.2. *A LS-manifold admitting a qsnmc is ϑ – Concircularly flat if and only if the scalar curvature $\bar{r}$ with respect to the qsnmc is equal to $2n(n-1)$.*

Proof. Combining (3) and (8), it follows that

$$\bar{C}(X,Y)Z = R(X,Y)Z + \Lambda(Z)[\Lambda(X)Y - \Lambda(Y)X] - [\Lambda(X)g(Y,Z) - \Lambda(Y)g(X,Z)]\vartheta - \frac{\bar{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{23}$$

Putting $Z=\vartheta$ in (23) and using (11), we have

$$\bar{C}(X,Y)\vartheta = R(X,Y)\vartheta + [-\Lambda(X)Y+\Lambda(Y)X] + \frac{\bar{r}}{n(n-1)}[-\Lambda(X)Y+\Lambda(Y)X]. \tag{24}$$

Using (20) and (24), we get

$$\bar{C}(X,Y)\vartheta = \left[2+\frac{\bar{r}}{n(n-1)}\right]R(X,Y)\vartheta. \tag{25}$$

If $\bar{C}(X,Y)\vartheta = 0$, then $\bar{r}=-2n(n-1)$ or $R(X,Y)\vartheta = \Lambda(Y)X-\Lambda(X)Y=0$, implies that $\Lambda(X)=0$ which is not possible.

Conversely, if $\bar{r}=-2n(n-1)$, then from (25), it follows that $\bar{C}(X,Y)\vartheta=0$.

This completes the proof.

Theorem 4.3. *If LS-manifolds satisfying $\bar{R}(\vartheta,Y)\cdot\bar{C}=0$ with respect to a qsnmc, then the manifold is an Λ-Einstein manifold with respect to the qsnmc and the scalar curvature $\bar{r}$ with respect to the qsnmc is a negative constant.*

Proof. In order to prove the Theorem, we first state following Lemma.

Lemma 4.4. [9] *If a LS-manifold is semi-symmetric with respect to the qsnmc, then the manifold is an Λ-Einstein manifold with respect to the qsnmc and the scalar curvature $\bar{r}$ with respect to the qsnmc is a negative constant.*

From the definition of concircular curvature tensor, it follows that

$$\bar{R}(X,Y)\cdot\bar{C} = \bar{R}(X,Y)\cdot\bar{R}.$$

Thus using Lemma 4.4, we obtain Theorem 4.3.

5. ϒ-Concircularly Flat Lorentzian Para-Sasakian Manifold with Respect to the Quarter-Symmetric Non Metric Connection

Definition 5.1. *A LS-manifold is said to be $\Upsilon-$Concircularly flat with respect to the qsnmc if*

$$\tilde{\bar{C}}(\Upsilon X,\Upsilon Y,\Upsilon Z,\Upsilon U)=0,$$

where X, Y, Z, $U\in\chi(M)$.

Definition 5.2. *A LS-manifold is said to be an $\Lambda-$Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form*

$$S(X,Y)=ag(X,Y)+b\Lambda(X)\Lambda(Y),$$

where a and b are smooth functions on the manifold.

Theorem 5.3. *If a LS-manifold admitting a qsnmc is Υ-Concircularly flat, then the manifold with respect to the qsnmc is an $\Lambda-$Einstein manifold.*

Proof. In view of (9) yields

$$\tilde{\bar{C}}(X,Y,Z,U) = \tilde{\bar{R}}(X,Y,Z,U) - \frac{\bar{r}}{n(n-1)}[g(Y,Z)g(X,U)-g(X,Z)g(Y,U)], \tag{26}$$

where $\tilde{\bar{C}}(X,Y,Z,U)=g(\bar{C}(X,Y)Z,U)$ and $\tilde{\bar{R}}(X,Y,Z,U)=g(\bar{R}(X,Y)Z,U)$.

Now putting $X = \Upsilon X,\ Y = \Upsilon Y,\ Z = \Upsilon Z,\ U = \Upsilon U$ in (26) and using (11) and (12), we get

$$\tilde{\bar{C}}(\Upsilon X,\Upsilon Y,\Upsilon Z,\Upsilon U) = \tilde{\bar{R}}(\Upsilon X,\Upsilon Y,\Upsilon Z,\Upsilon U) - \frac{\bar{r}}{n(n-1)}[g(\Upsilon Y,\Upsilon Z)g(\Upsilon X,\Upsilon U) - g(\Upsilon X,\Upsilon Z)g(\Upsilon Y,\Upsilon U)]. \tag{27}$$

Let $\{e_1, e_2, ..., e_{n-1}, \vartheta\}$ be a local orthonormal basis of vector fields in M, then $\{\Upsilon e_1, \Upsilon e_2, ..., \Upsilon e_{n-1}, \vartheta\}$ is also a local orthonormal basis. Putting $X = U = e_i$ in (27) and summing over $i = 1$ to $n-1$, we obtain

$$\sum_{i=1}^{n-1} \tilde{\bar{C}}(\Upsilon e_i,\Upsilon Y,\Upsilon Z,\Upsilon e_i) = \bar{S}(\Upsilon Y,\Upsilon Z) - \frac{(n-2)\bar{r}}{n(n-1)} g(\Upsilon Y,\Upsilon Z). \tag{28}$$

Using (13) and (22) in (28), we have

$$\sum_{i=1}^{n-1} \tilde{\bar{C}}(\Upsilon e_i,\Upsilon Y,\Upsilon Z,\Upsilon e_i) = \bar{S}(Y,Z) - g(Y,Z) - (n-2)\Lambda(Y)\Lambda(Z) - \frac{(n-2)\bar{r}}{n(n-1)}[g(Y,Z) + \Lambda(Y)\Lambda(Z)]. \tag{29}$$

By virtue of (5) and (29), we have

$$\sum_{i=1}^{n-1} \tilde{\bar{C}}(\Upsilon e_i,\Upsilon Y,\Upsilon Z,\Upsilon e_i) = \bar{S}(Y,Z) - \left[1 + \frac{(n-2)\bar{r}}{n(n-1)}\right] g(Y,Z) - \left[(n-2) + \frac{(n-2)\bar{r}}{n(n-1)}\right]\Lambda(Y)\Lambda(Z). \tag{30}$$

If $\sum_{i=1}^{n-1} \tilde{\bar{C}}(\Upsilon e_i,\Upsilon Y,\Upsilon Z,\Upsilon e_i) = 0$, then

$$\bar{S}(Y,Z) = \left[1 + \frac{(n-2)\bar{r}}{n(n-1)}\right] g(Y,Z) + \left[(n-2) + \frac{(n-2)\bar{r}}{n(n-1)}\right]\Lambda(Y)\Lambda(Z),$$

or

$$\bar{S}(Y,Z) = a g(Y,Z) + b\Lambda(Y)\Lambda(Z),$$

where $a = \left[1 + \frac{(n-2)\bar{r}}{n(n-1)}\right]$ and $b = \left[(n-2) + \frac{(n-2)\bar{r}}{n(n-1)}\right]$.
From which it follows that the manifold is an Λ −Einstein manifold with respect to the qsnmc. Hence, proof of the Theorem 5.3 is complete.

6. Lorentzian Para-Sasakian Manifold Satisfying $\bar{C} \cdot \bar{S} = 0$ with Respect to a Quarter-Symmetric Non Metric Connection

Theorem 6.1. *If LS-manifolds satisfying* $\bar{C} \cdot \bar{S} = 0$ *with respect to a qsnmc, then the manifold is an* Λ*-Einstein manifold with respect to a qsnmc.*

Proof. We consider LS-manifolds with respect to a qsnmc $\bar{\nabla}$ satisfying the curvature condition $\bar{C} \cdot \bar{S} = 0$. Then

$$(\bar{C}(X,Y) \cdot \bar{S})(U,V) = 0.$$

So,

$$\bar{S}(\bar{C}(X,Y)U,V)+\bar{S}(U,\bar{C}(X,Y)V)=0. \tag{31}$$

Putting $X=\vartheta$ in (31) and using (10), we get

$$\left(-\frac{\bar{r}}{n(n-1)}\right)[g(Y,U)\bar{S}(\vartheta,V)+g(Y,V)\bar{S}(\vartheta,U)]+\left(\frac{\bar{r}}{n(n-1)}+2\right)[\Lambda(U) \tag{32}$$

$$\bar{S}(Y,V)+\Lambda(V)\bar{S}(Y,U)]-2\Lambda(Y)\Lambda(U)\bar{S}(\vartheta,V)-2\Lambda(Y)\Lambda(V)\bar{S}(\vartheta,U)=0.$$

Again Putting $U=\vartheta$ in (32), implies that

$$\left(-\frac{\bar{r}}{n(n-1)}\right)[-2(n-1)\Lambda(Y)\Lambda(V)-2(n-1)g(Y,V)]+\left(\frac{\bar{r}}{n(n-1)}+2\right)[-\bar{S}(Y,V)$$
$$+2(n-1)\Lambda(Y)\Lambda(V)]+8(n-1)\Lambda(Y)\Lambda(V)=0.$$
$$\bar{S}(Y,V)=\left[\frac{-2(n-1)\bar{r}}{\bar{r}+2n(n-1)}\right]g(Y,V)+\left[\frac{-4n(n-1)^2}{\bar{r}+2n(n-1)}\right]\Lambda(Y)\Lambda(V)$$

Therefore, $\bar{S}(Y,V)=ag(Y,V)+b\Lambda(Y)\Lambda(V)$ where $a=\frac{-2(n-1)\bar{r}}{\bar{r}+2n(n-1)}$ and $b=\frac{-4n(n-1)^2}{\bar{r}+2n(n-1)}$.

This means that the manifold is an Λ-Einstein manifold with respect to the qsnmc.

7. Example

In this section, we construct an example on LS-manifold with respect to the qsnmc $\bar{\nabla}$, which verify the results of section three and section four. We consider the 5-dimensional manifold $(x,y,z,u,v)\in\mathbb{R}^5$, where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$. We choose the vector fields

$$\sigma_1=-\frac{\partial}{\partial x},\sigma_2=\sigma^{-x}\frac{\partial}{\partial y},\sigma_3=\sigma^{-x}\frac{\partial}{\partial z},\sigma_4=\sigma^{-x}\frac{\partial}{\partial u},\sigma_5=\sigma^{-x}\frac{\partial}{\partial v},$$

which are linearly independent at each point of *M*. Let *g* be the Lorentzian metric defined by

$$g(\sigma_1,\sigma_1)=-1,\ g(\sigma_2,\sigma_2)=1,(\sigma_3,\sigma_3)=1,\ g(\sigma_4,\sigma_4)=1,(\sigma_5,\sigma_5)=1,$$

and $g(\sigma_i,\sigma_j)=0$ if $i\neq j$.

Let Υ be the (1,1)-tensor field and $-\Lambda(X)=g(X,\sigma_1=\vartheta)$, be a 1-form defined by

$$\Upsilon\sigma_1=0,\ \Upsilon\sigma_2=-\sigma_3,\ \Upsilon\sigma_3=\sigma_2,\ \Upsilon\sigma_4=-\sigma_5,\ \Upsilon\sigma_5=-\sigma_4.$$

Using the linearity of Υ and *g*, we obtain

$$\Upsilon^2X=-X+\Lambda(X)\vartheta,\ \Lambda(\vartheta)=-1,\Lambda(X)=-g(X,\vartheta)$$

and

$$g(\Upsilon X,\Upsilon Y)=g(X,Y)+\Lambda(X)\Lambda(Y),$$

for any vector fields $X,Y\in\Gamma(TM)$. Thus for $\sigma_1=\vartheta$, the structure $(\Upsilon,\vartheta,\Lambda,g)$ defines an almost para-contact metric structure on *M*. Then, we have

$$[\sigma_1,\sigma_2]=\sigma_2,[\sigma_1,\sigma_3]=\sigma_3,[\sigma_1,\sigma_4]=\sigma_4,[\sigma_1,\sigma_5]=\sigma_5,$$
$$[\sigma_2,\sigma_3]=[\sigma_2,\sigma_4]=0,[\sigma_2,\sigma_5]=[\sigma_3,\sigma_4]=[\sigma_3,\sigma_5]=[\sigma_4,\sigma_5]=0.$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula, which is given by

$$2g(\nabla_X Y,Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]).$$

Using Koszul's formula, we obtain the following:

$$\begin{aligned}
&\nabla_{\sigma_1}\sigma_1 = 0, \nabla_{\sigma_1}\sigma_2 = 0, \nabla_{\sigma_1}\sigma_3 = 0, \nabla_{\sigma_1}\sigma_4 = 0, \nabla_{\sigma_1}\sigma_5 = 0,\\
&\nabla_{\sigma_2}\sigma_1 = -\sigma_2, \nabla_{\sigma_2}\sigma_2 = \sigma_1, \nabla_{\sigma_2}\sigma_3 = 0, \nabla_{\sigma_2}\sigma_4 = 0, \nabla_{\sigma_2}\sigma_5 = 0,\\
&\nabla_{\sigma_3}\sigma_1 = -\sigma_3, \nabla_{\sigma_3}\sigma_2 = 0, \nabla_{\sigma_3}\sigma_3 = \sigma_1, \nabla_{\sigma_3}\sigma_4 = 0, \nabla_{\sigma_3}\sigma_5 = 0,\\
&\nabla_{\sigma_4}\sigma_1 = -\sigma_4, \nabla_{\sigma_4}\sigma_2 = 0, \nabla_{\sigma_4}\sigma_3 = 0, \nabla_{\sigma_4}\sigma_4 = \sigma_1, \nabla_{\sigma_4}\sigma_5 = 0,\\
&\nabla_{\sigma_5}\sigma_1 = -\sigma_5, \nabla_{\sigma_5}\sigma_2 = 0, \nabla_{\sigma_5}\sigma_3 = 0, \nabla_{\sigma_5}\sigma_4 = 0, \nabla_{\sigma_5}\sigma_5 = \sigma_1.
\end{aligned}$$

In view of the above relations, we see that

$$\begin{aligned}
(\nabla_X \Upsilon)Y &= -g(X,Y)\vartheta - \Lambda(Y)X,\\
\nabla_X \vartheta &= -\Upsilon X,
\end{aligned}$$

for all $X,Y \in \Gamma(TM)$ and $\vartheta = \sigma_1$.

Therefore the manifold is a LS-manifold with the structure $(\Upsilon,\vartheta,\Lambda,g)$. Using (1) in above equations, we obtain

$$\begin{aligned}
&\bar{\nabla}_{\sigma_1}\sigma_1 = 0, \bar{\nabla}_{\sigma_1}\sigma_2 = 0, \bar{\nabla}_{\sigma_1}\sigma_3 = 0, \bar{\nabla}_{\sigma_1}\sigma_4 = 0, \bar{\nabla}_{\sigma_1}\sigma_5 = 0,\\
&\bar{\nabla}_{\sigma_2}\sigma_1 = -2\sigma_2, \bar{\nabla}_{\sigma_2}\sigma_2 = 2\sigma_1, \bar{\nabla}_{\sigma_2}\sigma_3 = 0, \bar{\nabla}_{\sigma_2}\sigma_4 = 0, \bar{\nabla}_{\sigma_2}\sigma_5 = 0,\\
&\bar{\nabla}_{\sigma_3}\sigma_1 = -2\sigma_3, \bar{\nabla}_{\sigma_3}\sigma_2 = 0, \bar{\nabla}_{\sigma_3}\sigma_3 = 2\sigma_1, \bar{\nabla}_{\sigma_3}\sigma_4 = 0, \bar{\nabla}_{\sigma_3}\sigma_5 = 0,\\
&\bar{\nabla}_{\sigma_4}\sigma_1 = -2\sigma_4, \bar{\nabla}_{\sigma_4}\sigma_2 = 0, \bar{\nabla}_{\sigma_4}\sigma_3 = 0, \bar{\nabla}_{\sigma_4}\sigma_4 = 2\sigma_1, \bar{\nabla}_{\sigma_4}\sigma_5 = 0,\\
&\bar{\nabla}_{\sigma_5}\sigma_1 = -2\sigma_5, \bar{\nabla}_{\sigma_5}\sigma_2 = 0, \bar{\nabla}_{\sigma_5}\sigma_3 = 0, \bar{\nabla}_{\sigma_5}\sigma_4 = 0, \bar{\nabla}_{\sigma_5}\sigma_5 = 2\sigma_1.
\end{aligned}$$

Now, we can easily obtain the non-zero components of the curvature tensors R as follows:

$$\begin{aligned}
&R(\sigma_1,\sigma_2)\sigma_1 = -\sigma_2, (\sigma_1,\sigma_2)\sigma_2 = \sigma_1, R(\sigma_1,\sigma_3)\sigma_1 = -\sigma_3, R(\sigma_1,\sigma_3)\sigma_3 = \sigma_1,\\
&R(\sigma_1,\sigma_4)\sigma_1 = -\sigma_4, (\sigma_1,\sigma_4)\sigma_2 = \sigma_1, R(\sigma_1,\sigma_5)\sigma_1 = -\sigma_5, R(\sigma_1,\sigma_5)\sigma_3 = \sigma_1,\\
&R(\sigma_2,\sigma_3)\sigma_1 = -\sigma_3, (\sigma_2,\sigma_3)\sigma_2 = \sigma_2, R(\sigma_2,\sigma_4)\sigma_1 = -\sigma_4, R(\sigma_2,\sigma_4)\sigma_3 = \sigma_2,\\
&R(\sigma_2,\sigma_5)\sigma_1 = -\sigma_5, (\sigma_2,\sigma_5)\sigma_2 = \sigma_2, R(\sigma_3,\sigma_4)\sigma_1 = -\sigma_4, R(\sigma_3,\sigma_4)\sigma_3 = \sigma_3,\\
&R(\sigma_3,\sigma_5)\sigma_1 = -\sigma_5, (\sigma_3,\sigma_5)\sigma_2 = \sigma_3, R(\sigma_4,\sigma_5)\sigma_1 = -\sigma_5, (\sigma_4,\sigma_5)\sigma_3 = \sigma_4,
\end{aligned}$$

and

$$\begin{aligned}
&\bar{R}(\sigma_1,\sigma_2)\sigma_1 = -2\sigma_2, \bar{R}(\sigma_1,\sigma_2)\sigma_2 = 2\sigma_1, \bar{R}(\sigma_1,\sigma_3)\sigma_1 = -2\sigma_3, \bar{R}(\sigma_1,\sigma_3)\sigma_3 = 2\sigma_1,\\
&\bar{R}(\sigma_1,\sigma_4)\sigma_1 = -2\sigma_4, \bar{R}(\sigma_1,\sigma_4)\sigma_2 = 2\sigma_1, \bar{R}(\sigma_1,\sigma_5)\sigma_1 = -2\sigma_5, \bar{R}(\sigma_1,\sigma_5)\sigma_3 = 2\sigma_1,\\
&\bar{R}(\sigma_2,\sigma_3)\sigma_1 = -2\sigma_3, \bar{R}(\sigma_2,\sigma_3)\sigma_2 = 2\sigma_2, \bar{R}(\sigma_2,\sigma_4)\sigma_1 = -2\sigma_4, \bar{R}(\sigma_2,\sigma_4)\sigma_3 = 2\sigma_2,\\
&\bar{R}(\sigma_2,\sigma_5)\sigma_1 = -2\sigma_5, \bar{R}(\sigma_2,\sigma_5)\sigma_2 = 2\sigma_2, \bar{R}(\sigma_3,\sigma_4)\sigma_1 = -2\sigma_4, \bar{R}(\sigma_3,\sigma_4)\sigma_3 = 2\sigma_3,\\
&\bar{R}(\sigma_3,\sigma_5)\sigma_1 = -2\sigma_5, \bar{R}(\sigma_3,\sigma_5)\sigma_2 = 2\sigma_3, \bar{R}(\sigma_4,\sigma_5)\sigma_1 = -2\sigma_5, \bar{R}(\sigma_4,\sigma_5)\sigma_3 = 2\sigma_4.
\end{aligned}$$

With the help of the above curvature tensors with respect to the qsnmc we find the Ricci tensors S as follows:

$$\bar{S}(\sigma_1,\sigma_1) = \bar{S}(\sigma_2,\sigma_2) = \bar{S}(\sigma_3,\sigma_3) = \bar{S}(\sigma_4,\sigma_4) = \bar{S}(\sigma_5,\sigma_5) = 8.$$

Also it follows that the scalar curvature tensor with respect to the quarter-symmetric metric connection is $\bar{r} = 40$.

Let X, Y, Z and U be any four vector fields given by

$$X = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 + a_4\sigma_4 + a_5\sigma_5, = b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3 + b_4\sigma_4 + b_5\sigma_5$$
$$Z = c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_3 + c_4\sigma_4 + c_5\sigma_5, = d_1\sigma_1 + d_2\sigma_2 + d_3\sigma_3 + d_4\sigma_4 + d_5\sigma_5$$

where a_i, b_i, c_i, d_i, for all $i = 1,2,3,4,5$ are all non-zero real numbers. Using the above curvature tensors and the scalar curvature tensors of the qsnmc, we have

$$\bar{C}(X,Y)\vartheta = (2\sigma_2 - 2\sigma_2)a_1b_1 + (2\sigma_5 - 2\sigma_5)a_1b_5 + (2\sigma_4 - 2\sigma_4)a_1b_4 + (2\sigma_3 - 2\sigma_3)a_1b_3 = 0,$$

which verifies the result of Section three.

Now, we see that the Υ-Concircularly flat with respect to the qsnmc from the above relations as follow:

$$\begin{aligned}\bar{C}(\Upsilon X, \Upsilon Y, \Upsilon Z, \Upsilon U) &= 2a_2b_3(c_2d_3 - c_3d_2) + 2a_2b_5(c_2d_5 - c_5d_2) + 2a_3b_4(c_3d_4 - c_4d_3) \\ &\quad + 2a_4b_5(c_4d_5 - c_5d_4) + 2a_2b_4(c_2d_4 - c_4d_2) + 2a_3b_5(c_3d_5 - c_5d_3) \\ &= 0\end{aligned}$$

Hence LS-manifolds will be Υ-Concircularly flat with respect to the quarter-symmetric metric connections if $\frac{c_2}{d_2} = \frac{c_3}{d_3} = \frac{c_4}{d_4} = \frac{c_5}{d_5}$. The above arguments tell us that the 5-dimensional LS-manifold with respect to the qsnmc under consideration agrees with the section five.

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