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Tor–prime and Strongly *Tor*–prime submodules and modules

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Abstract

The concepts of *Tor*–prime and Strongly *Tor*–prime submodules are introduced and investigated. A proper submodule N is called *Tor*–prime submodule (resp. Strongly *Tor*-prime), if $rm \in N$ (resp. $((N+Rx):Tor(M))y \subseteq N$), then $m \in N$ or $rTor(M) \subseteq N$ (resp. $x \in N$ or $y \in N$). A proper submodule P is *Tor*–prime submodule of M if and only if P_S is a *Tor*–prime (res. Strongly *Tor*–prime) submodule in M_S, where S is a prime ideal of R. A finitely generated module M is Noetherian if and only if every *Tor*–prime submodule of M is finitely generated. Furthermore, the Cohen Theorem can be generalized for these new classes of submodules.

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1. Introduction

A proper submodule N is called prime if $\frac{M}{N}$ is a prime module. In the other word, if $rm \in N$, then either $m \in N$ or $rM \subseteq N$ for any $r \in R, m \in M$ [1]. There has been a great deal of studding on prime submodules and their generalizations. For more knowledge about prime submodules, we recommend to see the references [1–8]. The notion of prime submodule plays an important role in the theory of rings and modules. Let N be a proper submodule of an R-module M. The module M is prime if and

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only if the zero submodule of M is prime. The submodule N of M is said to be Strongly prime, if whenever $((N+Rx):M)y \subseteq N$, then $x \in N$ or $y \in N$ [3]. It is well-known that every strongly prime is submodule prime. Also, N is called weakly prime submodule if $r \in R$ and $m \in M$ such that $0 \neq mr \in N$, then either $m \in N$ or $rM \subseteq N$ and N is called an n-almost prime submodule of M, if $r \in R$ and $m \in M$ such that $rm \in N-(N:M)^n N$, then either $m \in N$ or $rM \subseteq N$ [3]. Bataineh [3] proved that N is n-almost prime submodule of M if and only if $\frac{N}{(N:M)^n N}$ is a weakly prime submodule in $\frac{M}{(N:M)^n N}$.

The notion of $Tor-$prime and strongly $Tor-$prime submodules are introduce as a generalization of prime and Strongly prime submodules. A proper submodule N is called $Tor-$prime (resp. $Tor-$strongly prime), if $rm \in N$ (resp. $((N+Rx):Tor(M))y \subseteq N$), then $m \in N$ or $rTor(M) \subseteq N$ (resp. $x \in N$ or $y \in N$). A module is called $Tor-$prime, if zero submodule is $Tor-$prime. If M is a nonzero R-module. Then M is $Tor-$prime module if and only if $Ann(N) = Ann(Tor(M))$, for every nonzero submodule N of M. Let $M = \bigoplus_{i=1}^{n} M_i$. Then M_i is a $Tor-$prime submodule of M if and only if M_i is a $Tor-$prime module, for each $i = 1,2,\ldots,n$. Every prime (resp. strongly prime) submodule is $Tor-$prime (resp. $Tor-$ strongly prime) submodule. But the converse need not be true in general, see Example 2.2. Suppose that M is a torsion module, then every $Tor-$ prime submodule is prime submodule. This means that for a torsion module, $Tor-$prime property is a necessary and sufficient condition for a prime submodule. Suppose that M is a finitely generated R-module. Then M is a Noetherian module if and only if every $Tor-$ prime submodule of M is finitely generated. Suppose that P is a proper submodule of M and S is a prime(or maximal) ideal of R such that $P_S \neq M_S$. Then P is a $Tor-$prime (res. Strongly $Tor-$prime) submodule of M if and only if P_S is a $Tor-$prime (res. Strongly $Tor-$prime) submodule in M_S. And consequently, M is a $Tor-$prime module if and only if M_S is a $Tor-$prime module. If P_1 is a strongly $Tor-$prime (res. $Tor-$semiprime or strongly $Tor-$semiprime) submodule of M_1. Then $P_1 \times M_2$ is a strongly $Tor-$prime (res. $Tor-$semiprime or strongly $Tor-$semiprime) submodule of M. As a result, if P_1 and P_2 are $Tor-$prime submodules of M_i. Then $P_1 \times P_2$ is a $Tor-$prime submodule of M.

Unless stated otherwise, all rings are associative and have identity, all modules are unital left R-modules. If N is a submodule of M, then $(N:M) = \{r \in R : rM \subseteq N\}$ [9]. We focus on the torsion elements of M and we denote $Tor(M)$ for the set of all torsion elements of M. A module M is called torsion free if $Tor(M) = 0$, while M is said to be torsion module if $Tor(M) = M$. If R is an integral domain, then $Tor(M)$ is a submodule of M and $\frac{M}{Tor(M)}$ is a torsion free module. If R is not integral domain, then $Tor(M)$ need not be submodule and $\frac{M}{Tor(M)}$ need not be torsion free while $Tor(M)$ is submodule. We consider that all the $Tor(M)$ is torsionable in this paper [10]. If S is a multiplicative closed system, then M_S is an R_S-module which is called the localization (quotient) of M at S [11]. If P is a prime ideal in R, then $R-S$ forms a multiplicative closed system, then we denote M_P for the localization of M at $R-S$.

2. *Tor*–prime and Strongly *Tor*–prime submodules

In this section, we define *Tor*–prime and Strongly *Tor*–prime submodules as generalizations of prime and Strongly prime submodules. A proper submodule N of M is said to be prime (resp. Strongly prime), if whenever $rm \in N$ (resp. $((N+Rx):M)y \subseteq N$), then $m \in N$ or $rM \subseteq N$ (resp. $x \in N$ or $y \in N$) [2, 3, 12]. We start with the following definition:

Definition 2.1: *Let M be an R - module, and $Tor(M) = \{x \in M; \exists 0 \neq r \in R$ such that $rx = 0\}$. Then a proper submodule N is said to be a Tor–prime submodule, if whenever $r'm \in N$ for $r' \in R$ and $m \in M$, then $m \in N$ or $r'Tor(M) \subseteq N$.*

Example 2.2:

1. If R is an integral domain, then for any R-module M, we can show that $Tor(M) \neq M$ is a $Tor-$prime submodule.
2. If M is a projective (resp. flat) R-module and N is a submodule of M. Then one can easily obtain that N is a *Tor*–prime submodule.
3. Z is a Z-module and $N = \langle 4 \rangle$ is a proper submodule of Z and $Tor(M) = \{0\}$, then N is a *Tor*–prime. But N is not prime submodule since $2.2 = 4 \in \langle 4 \rangle$ where $2 \notin N$ and $2Z \nsubseteq N$ ssince $6 \in 2Z$ but $6 \notin \langle 4 \rangle$.
4. $Z_8 = \{0,1,2,3,4,5,6,7\}$ is a Z_8-module and $N_1 = \{0,2,4,6\}$ and $N_2 = \{0,4\}$ is a proper submodule of Z_8 and $Tor(M) = \{0,2,4,6\}$, then N_1 is prime and $Tor-$prime and N_2 is T-prime but not a prime submodule of Z_8. 5- Consider $Z_{12} = \{0,1,2,\ldots,11\}$ as a Z-module. Then $Tor(Z_{12}) = Z_{12}$ and $N = \{0,6\}$ is a submodule of Z_{12} which it is not a *Tor*–prime submodule in Z_{12}. Since $2.3 = 6 \in N$, but $3 \notin N$ and 2 . $Tor(Z_{12}) = \{0,2,4,6,8,10\} \nsubseteq N$. Hence the above definition in nontrivial.

Proposition 2.3: *Every prime submodule is a Tor–prime submodule.*

Proof. Let N be a prime submodule and $rm \in N$. Then $m \in N$ or $rM \subseteq N$, this implies that N is *Tor*–prime.

The converse of Proposition 2.3 need not be true in general, from Example 2.2 (1 and 2), we explain this fact. Suppose that M is a torsion module, then every *Tor*–prime submodule is a prime submodule. This means that for a torsion module, *Tor*–prime property is a necessary and sufficient condition for a prime submodule.

Theorem 2.4: *Every submodule of a torsion free module is Tor–prime submodule.*

Proof. Let M be a torsion free module and N be a submodule of M. If $rm \in N$ and $m \notin N$, then it is obvious that $rTor(M) \subseteq N$. Hence N is a $Tor-$prime submodule, for any arbitrary submodule N of M.

If M is a module over an integral domain. Then from Theorem 4 we conclude that every submodule of the quotient module $\dfrac{M}{Tor(M)}$ is a *Tor*–prime submodule. That means the zero submodule $(Tor(M))$ is a *Tor*–prime submodule. Furthermore, in the following corollary we conduct that projective (flat) module is a *Tor*–prime module.

Corollary 2.5: *Every submodule of projective (flat) modules is a Tor–prime submodule.*

Theorem 2.6: *Let M and N be two R-modules and $f : M \to N$ be an epimorphism such that K is a $Tor-$prime submodule of M with $kerf \subseteq K$. Then $f(K)$ is a $Tor-$prime submodule of N.*

Proof. Let K be a $Tor-$prime submodule in M. If $f(K) = N$, then $f(K) = f(M)$. Now, for any $m \in M$, then $f(m) \in N = f(K)$, this implies that $f(m) = f(k)$, for some $k \in K$, so $m - k \in kerf$. We can write $m = m - k + k$, and this is an element of $K + kerf$, so $M = K + kerf$ this contradict the assumption. Hence, $f(K) \subset N$. If $rm' \in f(K)$, then there exists $m \in K$ such that $m' = f(m)$. Since, $rm \in K$, then $m \in K$ or $rTor(M) \subseteq K$. If $m \in K$, then $m' \in f(K)$ and we are done. If $m \notin K$, then $rTor(M) \subseteq K$, first we have to show the following: for any $x \in Tor(M)$, then there exists a nonzero element r in R such that $rx = 0$, this gives that $rf(x) = 0$ and since f is an epimorphism, then the converse is also true, in other hands we have $f(Tor(M)) = Tor(N)$. Now, suppose that $rn \in rTor(N)$, then there exists

$m_1 \in Tor(M)$, such that $f(m_1) = n$, since $rTor(M) \subseteq K$, thus we obtain $rn = rf(m_1) = f(rm_1) \in f(K)$. This completes the proof.

Proposition 2.7: *Proposition 7. If M and N are two R-modules and $f: M \to N$ be an epimorphism such that K' is a Tor –prime submodule of N with $f(M) \subseteq K'$. Then the inverse image of K' is a Tor – prime submodule of M.*

Proof. Suppose that $f^{-1}(K') = M$, then for any $x \in M$, we have $x \in f^{-1}(K')$, this means that $f(x) \in K'$ and it is a contradiction to the hypothesis. Let $rx \in f^{-1}(K')$. Then $rf(x) \in K'$, since K' is a Tor –prime submodule, so $f(x) \in K'$ or $rTor(N) \subseteq K'$. If $f(x) \in K'$, then we are done, if $f(x) \notin K'$, then $rTor(N) \subseteq K'$. Now, suppose that $rm \in rTor(M)$, for some $m \in Tor(M)$. Then $f(m) \in f(Tor(M)) \subseteq Tor(N)$. So, $f(rm) = rf(m) \in rTor(N) \subseteq K'$. Hence $rm \in f^{-1}(K')$.

As a consequence of Theorem 6. If N is a Tor –prime submodule of M and K is any submodule of M, which containing N. Then $\frac{K}{N}$ is again form a Tor –prime submodule of $\frac{M}{N}$. Suppose that M is a nonzero R-module. Then M is said to be Tor –prime module if $\{0\}$ is Tor – prime submodule in M, and M is said to be almost fully Tor –prime module, if all proper submodules are Tor – prime submodule. In the next theorem, we investigate the characteristic of Tor –prime module in terms of annihilator of its nonzero submodules.

Theorem 2.8: *If M is a nonzero R-module. Then M is a Tor –prime module if and only if $Ann(N) = Ann(Tor(M))$, for every nonzero submodule N of M.*

Proof. Let N be a nonzero Tor – prime submodule in M and $r \in Ann(N)$. Then there exists a nonzero element x of N such that $rx = 0$, since $\{0\}$ is a Tor –prime submodule, then $rTor(N) \subseteq \{0\}$. Hence, $r \in Ann(Tor(M))$. In a similar argument we can show that $Ann(Tor(M)) \subseteq Ann(N)$. For the converse, suppose that $rm = 0$, for $r \in R$ and $0 \neq m \in Tor(M)$, then by our assumption we have $Ann(Rm) = Ann(Tor(M))$, so $rTor(M) \subseteq \{0\}$. Hence $\{0\}$ is a Tor – prime submodule in M.

If M is a Tor –prime module, then from Theorem 6 we conclude that every direct summand of M is again Tor –prime module. In the following theorem we discuss this fact in general.

Theorem 2.9: *Let M be an R-module and $M = \bigoplus_{i=1}^{n} M_i$. Then M_i is a Tor –prime submodule of M if and only if M_i is a Tor – prime module, for each $i = 1, 2, \ldots, n$.*

Proof. Let M_i be a Tor –prime submodule of M, for $i = 1, 2, \ldots, n$. Then $\{0\}$ is a direct summand in each submodule M_i, thus by Theorem 6 we obtain $\{0\}$ is a Tor –prime submodule in M_i as required. Conversely, suppose that M_i is a Tor –prime module for each i. Then the epimorphism $f_i : M_i \to M$ defined by $f_i(m_i) = (m_i, 0, \ldots, 0)$ conducts that M_i is a Tor –prime as a submodule of M.

In the following corollary, we obtain a characterization of Tor –prime module. This result is an application of Theorem 2.9.

Corollary 2.10: *If M is an R-module. Then the following properties are equivalent:*

1. M is a Tor – prime module,
2. Every direct summand of M are Tor – prime submodule of M,
3. For all $0 \neq m \in M, Ann(m) = Ann(Tor(M))$.

Proof. It is a consequence of Theorem 9 and definition of Tor –prime submodule. It is similar to [1, Proposition 1.2].

In the next proposition we investigate relation between Tor –prime and almost fully Tor – prime modules.

Proposition 2.11: *Let M be a module. Then M is almost fully Tor –primary if and only if the following conditions hold:*

1. M is a Tor – prime module,
2. M is an indecomposable.
3. M must have exactly two nonzero indecomposable Tor – prime direct summands, say M_1 and M_2.

Proof. Suppose that M is an almost fully Tor –prime module. On contrary, if the conditions does not hold. Since M_i are submodules of M, then by our hypothesis M_i are Tor –prime submodules of M, for $i = 1,2$. So, M_i are Tor –prime modules, this implies that $Ann(x) = Ann(Tor(M_1))$ and $Ann(y) = Ann(Tor(M_2))$, for every $0 \neq x \in M_1$ and $0 \neq y \in M_2$. We have to show that $Ann(Tor(M_1)) \neq Ann(Tor(M_2))$. If $Ann(Tor(M_1)) = Ann(Tor(M_2))$, then $Ann(m) = Ann(Tor(M))$, for any $0 \neq x \in M$. Then M is a Tor – prime module, which is a contradiction. Without loss of generality, suppose that M_1 is a decomposable. Then there exist two nonzero submodules H and K such that $M_1 = H \oplus K$. This gives that $M = P \oplus Q \oplus M_2$ and $P \oplus Q$ is a Tor –prime submodule, that is $Ann(Tor(Q)) = Ann(Tor(M_2))$. Then $Ann(Tor(M_1)) = Ann(Tor(M_2))$, which is again contradiction. Hence M_1 and M_2 are indecomposable. It is easy to show that M_1 and M_2 are unique with the property (3). Conversely, if the conditions are hold, then it is obvious the submodules of M would be Tor –prime submodule as desired.

From [1, corollary 1.8] and Proposition 11, we obtain the following result.

Corollary 2.12: *Corollary 12. Let a module M be the direct summand of two simple submodules. Then M is Tor –prime or almost fully Tor –prime.*

Suppose that $\phi : S(M) \to S(M) \cup \{\phi\}$ is a function. Then a proper submodule P of M is said to be $(n-1,n)-\phi$-prime, if whenever, $a_1.a_2 \ldots a_{n-1}x \in P - \{\phi(P)\}$, where $a_1, a_2, \ldots, a_{n-1} \in R$ and $x \in M$, then $a_1 \cdot a_2 \ldots a_{i-1} \cdot a_{i+1} \ldots a_{n-1}x \in P$ or $a_1 \cdot a_2 \ldots a_{n-1}M \subseteq P$ [4]. In the following theorem we show that $(n-1,n)-\phi$-prime property is a necessary condition for Tor – prime property in a special case.

Theorem 2.13: *If a proper submodule P of M is $(n-1,n)-\phi$-prime, where $n = 1$ and $\phi(P) = \phi$. Then P is a Tor –prime submodule.*

Proof. It is obvious.

Definition 2.14: *Let $P \subset M$. Then P is called strongly Tor –prime submodule, if $((P + Rx) : Tor(M))y \subseteq P$, then $x \in P$ or $y \in P$. Also, P is strongly Tor –semiprime if $((P + Rx) : Tor(M))x \subseteq P$, then $x \in P$.*

If a proper submodule P of M is a strongly Tor –prime submodule. Then we claim that P is Tor – prime submodule. Suppose that P is not a Tor –prime submodule of M, then there exist $x \in M - \{P\}$ such that $rx \subseteq P$, whereas $rTor(M) \nsubseteq P$. This means that there exists $y \in Tor(M)$ such that $ry \notin P$, then $((P + Rx) : Tor(M))ry = r((P + Rx) : Tor(M))y \subseteq P$, and since P is a strongly Tor –prime, then $x \in P$ or $ry \in P$ in each case we arrive to a contradiction. Hence every strongly Tor –prime submodule is Tor –prime submodule, but the converse need not be true in general (See Example 2(3)). Since we have shown that $N = \langle 4 \rangle$ is a Tor –prime submodule of Z. Now, consider that $M = Z \times Z$ as a Z-module and $P = N \times N$. It is clear that P is a Tor – prime submodule of M and $((P + R(2,0)) : Tor(M))(2,0) \subseteq P(2,0) \subseteq P$, where $(2,0) \notin P$. In the next theorem, we discuss a condition for which the converse is hold.

Theorem 2.15: *Let N be a proper submodule of M. Then N is strongly Tor –prime if and only if it is strongly Tor –semiprime and Tor –prime.*

Proof. Let N be strongly Tor –prime and $((P + Rx) : Tor(M))x \subseteq P$, for $x \in M$. Since N is strongly Tor –prime, then we obtain $x \in P$, this means that N is strongly Tor –semiprime and by the above explanation. Also, N is $Tor-$prime. Conversely, suppose that N is strongly $Tor-$semiprime and $Tor-$ prime. If $((P+Rx)$:Tor $(M))y \subseteq P$, for $x, y \in M$, then there exists $r \in ((P+Rx):Tor(M))$ such that $ry \in P$, since P is a $Tor-$prime submodule, then $rTor(M) \subseteq P$ and $rTor(M) \subseteq P+Rx$. Hence $Rx \subseteq P$ as required.

The following diagram, explains the relations between these new classes of submodules and the classical classes:

Strongly prime $\Longrightarrow$ prime $\Longrightarrow$ $(n-1,n)-\phi$ $-$prime $\Longrightarrow$ strongly Tor $-$prime $\Longrightarrow$ Tor $-$prime

$\searrow$ $\searrow$ $\swarrow$ $\swarrow$ $\swarrow$

Strongly semiprime $\Longrightarrow$ semiprime $\Longrightarrow$ strongly Tor $-$semiprime $\Longrightarrow$ Tor $-$semiprime

In each case, the converse of the above diagram need not be true in general (see Example 2.2 and [3, 4, 7].

Proposition 2.16: *Let N be a $Tor-$prime submodule of M. Then N contain a minimal $Tor-$prime submodule of M.*

Proof. Suppose that F is the set of all $Tor-$prime submodules of M that contain in N. Since $N \in F$, then F is not empty set. So, by Zorn's Lemma F contains a minimal element with respect to the inclusion relation. Now, we have to show that any chain L in F has a lower bound in F. Let $Q = \cap_{U \in L} U$, where U is $Tor-$ primesubmodule in M. Then for any $r \in R, m \in M$ such that $rm \in U$, implies that $m \in U$ or $rTor(M) \subseteq U$. First, we want to show Q is a $Tor-$prime and it is contain in N. For this we suppose that $rm \in Q$, then $rm \in U$, for any $U \in L$. If $m \notin Q$, then there exist $U \in L$ such that $m \notin U$, but U is a $Tor-$prime submodule, then we obtain $rTor(M) \subseteq U$, for any $U \in L$. Hence Q is a $Tor-$prime. Since the elements of L are in F, then each elements are contained in N, this means that Q is also contained in N. Thus $Q \in F$ and it is a lower bound in F. By the same way as above we use Zorn's Lemma on the lower bound in F, then there exist a minimal element Q' among the $Tor-$ prime submodules containes in F and any $Tor-$ prime submodule contain in Q' is in F. Hence we obtain the fact that Q' is the minimal $Tor-$prime submodule which contained in N.

The following theorem demonstrates how the structure of a finely generated module is influenced by $Tor-$prime submodules. In addition, this theorem can be regarded as a generalization of the theorem of Cohen, a classic commutative algebra theorem.

Theorem 2.17: *Suppose that M is a finitely generated R-module. Then M is a Noetherian module if and only if every $Tor-$ prime submodule of M is finitely generated.*

Proof. The (if) part is clear. The (only if) part is a consequence of Theorem 2.16.

If R is a ring, M is an R-module and R_S is the ring of fraction of R at the multiplicatively closed set $S \subset R$. Then an R_S-module M_S is also an R-module which is constructed via the canonical mapping $\lambda_S : R \to R_S : rx = \lambda_S(r)x = \frac{r}{1}x$, for any $r \in R, m \in M$. It is well-known that every submodules of M_S has the form N_S, where N is a submodule of M. The R_S-module M_S constructed above is called the module of fractions with denominators in S (Localization of M at S) [11]. From [10] it can be seen that $Tor(M)_S = Tor\left(M_S\right)$. As a first result, we can show that there is a one to one correspondence between all $Tor-$ prime submodules P of M with $(P:Tor(M)) \cap S = \phi$ and the $Tor-$prime submodules of M_S. In the following theorem, we show that the properties of $Tor-$prime (semiprime), strongly $Tor-$prime (semiprime) can preserved under the localization.

Theorem 2.18: *Let P be a proper submodule of M, S is a multiplicative closed set of R such that $P_S \neq M_S$. Then the following statements are hold:*

1. If P is a $Tor-$prime (res. Strongly $Tor-$prime) submodule of M, then P_S is a $Tor-$prime (res. Strongly $Tor-$ prime) submodule in M_S.
2. If P is a $Tor-$semiprime (res. Strongly $Tor-$semiprime) submodule of M, then P_S is a $Tor-$semiprime (res. Strongly $Tor-$semiprime) submodule in M_S.

Proof. We prove the first part and the second part can be in similar argument. Let P be a $Tor-$prime submodule of M and $\frac{r}{s} \in R_S, \frac{m}{t} \in M_S$ such that $\frac{r}{s} \cdot \frac{m}{t} \in P_S$, where $r \in R, m \in M$ and $s,t \in S$. Then there exists $u \in S$ such that $u.r.m \in P$ and since P is a $Tor-$prime, then $m \in P$ or $rTor(M) \subseteq P$. Then by the properties of $Tor(M)$ we obtain that $\frac{m}{t} \in P_S$ or $\frac{r}{s}Tor(M_S) \subseteq P_S$. Hence P_S is a $Tor-$prime submodule in M_S. If P is strongly $Tor-$prime submodule of M, then by the same way we can show that P_S is strongly $Tor-$prime submodule of M_S.

If P_S is a $Tor-$prime submodule of M_S, then it is not necessary to P be $Tor-$prime submodule in M. Thus the converse of Theorem 18 need not be true in general. In the following corollary we investigate whether the converse is also hold.

Corollary 2.19: *Let P be a proper submodule of M, S is a prime(or maximal) ideal of R such that $P_S \neq M_S$. Then the following statements are hold:*

1. P is a $Tor-$prime (res. Strongly $Tor-$prime) submodule of M if and only if P_S is a $Tor-$prime (res. Strongly $Tor-$prime) submodule in M_S.
2. P is a $Tor-$semiprime (res. Strongly $Tor-$semiprime) submodule of M if and only if P_S is a $Tor-$ semiprime (res. Strongly $Tor-$semiprime) submodule in M_S.
3. M is a $Tor-$prime module if and only if M_S is a $Tor-$prime module.

Proof. It is similar to Theorem 2.18.

Consider that R_i is commutative ring with identity and M_i is an R_i-module, for $i = 1,2$. Suppose that $R = R_1 \times R_2$, then $M = M_1 \times M_2$ is an R-module and the submodules of M all are of the form $N_1 \times N_1$, where N_i is a submodule of M_i and $Tor(M) = Tor(M_1) \times Tor(M_2)$. In the following theorem we discuss the relation between $Tor-$prime (res. Strongly $Tor-$prime) submodules of M and $M_i, i = 1,2$.

Theorem 2.20: *Let P_1 be a $Tor-$prime submodule of M_1. Then $P_1 \times M_2$ is a $Tor-$prime submodule of M.*

Proof. Suppose that P_1 is a $Tor-$prime submodule of M_1 and $(r_1m_1, r_2m_2) \in P_1 \times M_2$, where $r_i \in R_i, m_i \in M_i$. Then $r_1m_1 \in P_1$ and since P_1 is a $Tor-$ prime, so we have $m_1 \in P_1$ or $r_1Tor(M_1) \subseteq P_1$. This means that $(m_1, m_2) \in P_1 \times M_2$ or $(r_1, r_2)Tor(M) = (r_1, r_2)Tor(M_1) \times Tor(M_2) \subseteq P_1 \times M_2$. Hence $P_1 \times M_2$ is a $Tor-$prime submodule of M.

We can easily follow the process of Theorem 20 to obtain the following properties: If P_1 is a strongly $Tor-$prime (res. $Tor-$semiprime or strongly $Tor-$semiprime) submodule of M_1. Then $P_1 \times M_2$ is a strongly $Tor-$prime (res. $Tor-$semiprime or strongly $Tor-$semiprime) submodule of M. As a result, if P_1 and P_2 are $Tor-$ prime submodules of M_i. Then $P_1 \times P_2$ is a $Tor-$prime submodule of M. Furthermore, in the next theorem we conduct that the converse of Theorem 20 is also hold.

Theorem 2.21: *Suppose that $N_1 \times N_2$ is a $Tor-$prime submodule of M. Then one of the N_i's must be $Tor-$prime submodule of M_i.*

Proof. Let $N_1 \times N_2$ be a $Tor-$prime submodule of M. If $N_1 = M_1$ and $N_2 = M_2$, then we have $N_1 \times N_2 = M$ which is impossible. Then $N_1 \neq M_1$ or $N_2 \neq M_2$. If N_2 is $Tor-$prime, then we are done, if not, suppose that $N_1 \neq M_1$ and $r_1m_1 \in N_1$, so $(r_1m_1, 0) \in N_1 \times N_2$ and since $N_1 \times N_2$ is a $Tor-$prime submodule in M, then $(m_1,0) \in N_1 \times N_2$ or $(r_1,0)Tor(M_1) \times Tor(M_2) \in N_1 \times N_2$. This implies that $m_1 \in N_1$ or $r_1 Tor(M_1) \subseteq N_1$. Hence N_1 is a $Tor-$prime submodule of M_1.

From above theorem, we conclude that if $N_1 \times N_2$ is a strongly $Tor-$prime (res. $Tor-$semiprime or strongly $Tor-$semiprime) submodule of M. Then one of the N_i 's must be strongly $Tor-$prime (res. $Tor-$semiprime or strongly $Tor-$semiprime) submodule of M_i. Finally, if $M = M_1 \times M_2 \times \ldots \times M_n$ is an R-module, where $R = R_1 \times R_2 \times \ldots \times R_n$. Such that M is $Tor-$prime module, then M_i is $Tor-$prime R_i-module, for all $i = 1,2$.

Example 2.22: *Suppose that $M = M_1 \times M_2 \times \ldots \times M_n$ is an R-module, where $R = R_1 \times R_2 \times \ldots \times R_n$. Such that R_i is integral domain for some i. Then $Tor(M_1 \times M_2 \times \ldots \times M_n)$ is a strongly $Tor-$prime (res. $Tor-$semiprime or strongly $Tor-$ semiprime) submodule of M.*

Conclusion

In this paper, we introduced the concepts $Tor-$prime and strongly $Tor-$ prime submodule as generalization of prime and strongly prime submodule. We investigate some characterizations and equivalent conditions of the described concepts. If M is a nonzero R-module. Then M is $Tor-$ prime module if and only if $Ann(N) = Ann(Tor(M))$, for every nonzero submodule N of M. Let $M = \bigoplus_{i=1}^{n} M_i$. Then M_i is a $Tor-$prime submodule of M if and only if M_i is a $Tor-$prime module, for each $i = 1,2,\ldots,n$. Every prime (resp. strongly prime) submodule is $Tor-$ prime (resp. $Tor-$ strongly prime) submodule. But the converse need not be true in general, see Example 2. Suppose that M is a torsion module, then every $Tor-$prime submodule is prime submodule. This means that for a torsion module, $Tor-$prime property is a necessary and sufficient condition for a prime submodule. Suppose that M is a finitely generated R-module. The new concepts $Tor-$ prime and strongly $Tor-$ prime submodules are essential in module theory. Because, they have new properties by which we can define and investigate new structures in module theory. For instance, the reader can investigate the impact of these concepts on different types of modules such as, multiplication, projective and injective modules.

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