Results in Nonlinear Analysis 6 (2023) No. 3, 97–106
https://doi.org/10.31838/rna/2023.06.03.014
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

(A, B)-ω-contraction mappings in menger spaces

Youssef Achtoun, Mohammed Lamarti Sefian, Ismail Tahiri

Department of Mathematics Computing and Physical Sciences, Normal Higher School, Abdelmalek Essaadi University, Martil, Morocco

Abstract

In the current work, we will focus on coincidence point and common fixed point property for a family of single mappings in Menger spaces. In order to realize our objective, we present the concept of (A,B)-ω-probabilistic contraction, and by utilizing these one, we will examine the common fixed point property for a family of mappings in Menger spaces. The related common fixed point property in fuzzy metric spaces is achieved as a result of our main finding. Finally, we will give some relatives results in ordinary metric spaces to illustrate the main theorem.

Mathematics Subject Classification (2010): 54E70, 54H25, 47S50, 47H10.

Key words and phrases: Common fixed point theorem, (A,B)-ω-probabilistic contraction, Menger spaces, Fuzzy metric spaces

1. Introduction

Actually, the fixed point theory in Menger space is a fundamental branch of mathematical analysis, and it has been at the core of numerous scientific research because of it's large fields of applications. Some of the important problems in fixed point theory in this type of spaces is to establish a common fixed point theorem for a pair of mappings or more under some specific contraction conditions, which was studied by many authors [1–3]. This problematic was connected with relations among mappings (commutativity, compatibility...). Several contraction mappings theorems for commuting and compatibles mappings have been shown in probabilistic metric spaces [4–7]. Mishra [5] has extended the concept of mappings compatibility (presented by Jungck [8] in ordinary metric spaces), later Singh and Jain [9] have defined the weakly compatible maps concept in Menger spaces.

Email addresses: achtoun44@outlook.fr (Youssef Achtoun)*; memdema@yahoo.fr (Mohammed Lamarti Sefian); istahiri@uae.ac.ma (Ismail Tahiri)

Received July 24, 2023; Accepted August 16, 2023; Online September 10, 2023

From the contraction condition suggested by Jachymski in [10] and the weakly compatible maps concept, we will examine the coincidence and common fixed point of three mappings in order to get this property for a family of mappings in Menger space. As an extend of these result we also obtain some corresponding common fixed point theorems in fuzzy metric spaces. Our results generalizes various fixed point theorem [8, 11, 12, 13, 14].

2. Preliminaries

Now we state some basic definitions and facts from Menger space.

Definition 2.1: *A map $\xi:[0,+\infty]\to[0,1]$ is called a distance distribution function if the following statements are verified:*

1. ξ is left continuous on $[0,+\infty]$,
2. ξ is non-decreasing,
3. $\xi(0)=0$ and $\xi(+\infty)=1$.

We denote by Δ^+ the class of all distance distribution functions. The subset $D^+\subset\Delta^+$ is the set $D^+=\left\{\xi\in\Delta^+:\lim\limits_{x\to\infty}\xi(x)=1\right\}$.

A specific element of D^+ is the Heavyside function ε_a defined as:

$$\varepsilon_a(x)=\begin{cases}0 & if \quad x\in[0,a],\\ 1 & if \quad x\in(a,+\infty].\end{cases}$$

Definition 2.2: [15] *Triangular norm, or t-norm for short, is a mapping $\mathcal{H}:[0,1]\times[0,1]\to[0,1]$ where for all $x_1,x_2,x_3\in[0,1]$ the following statements are satisfied:*

1. $\mathcal{H}(x_1,x_2)=\mathcal{H}(x_2,x_1)$;
2. $\mathcal{H}(x_1,\mathcal{H}(x_2,x_3))=\mathcal{H}(\mathcal{H}(x_1,x_2),x_3)$;
3. $\mathcal{H}(x_1,x_2)<\mathcal{H}(x_3,x_4)$ for $x_1\le x_3$ and $x_2\le x_4$;
4. $\mathcal{H}(x_1,1)=x_1$.

Among the basic examples of t-norms are:

$$\mathcal{H}_M(x_1,x_2)=Min(x_1,x_2),\ \mathcal{H}_P(x_1,x_2)=x_1.x_2 \text{ and } \mathcal{H}_L(x_1,x_2)=Max(x_1+x_2-1,0).$$

Let $\mathcal{H}$ be a given t-norm and $n\in\mathbb{N}$, then for all $x_1,x_2,\ldots,x_n\in[0,1]$ we write

$$\mathcal{H}^n(x_1,x_2,\ldots,x_n)=\begin{cases}1 & if \qquad n=0,\\ \mathcal{H}(\mathcal{H}^{n-1}(x_1,x_2,\ldots,x_{n-1}),x_n) & otherwise.\end{cases}$$

And for every $x\in[0,1]$, the operation $\mathcal{H}^n(x)$ is defined by

$$\mathcal{H}^n(x)=\begin{cases}1 & if \qquad n=0,\\ \mathcal{H}(\mathcal{H}^{n-1}(x),x) & otherwise.\end{cases}$$

Definition 2.3: [15] *We say that a t-norm $\mathcal{H}$ is of H-type if the family $(\mathcal{H}^n(x))_{n\in\mathbb{N}}$ is equi-continuous at the point $x=1$, that is*

$$\forall\varepsilon\in(0,1),\exists\lambda\in(0,1):t>1-\lambda\Rightarrow\mathcal{H}^n(t)>1-\varepsilon \quad for\ all\ n\ge1.$$

$\mathcal{H}_M$ is an example H-type t-norm, but $\mathcal{H}_L$ is not, for more details (see e.g. [15]).

We note that, if $\mathcal{H}$ is a t-norm of H-type, then $\sup\limits_{a<1}\mathcal{H}(a,a)=1$.

Definition 2.4: [15] *The triple* $(\mathcal{Z},\mathcal{F},\mathcal{H})$ *where* $\mathcal{Z}$ *is nonempty set,* $\mathcal{F}$ *is a function from* $\mathcal{Z}\times\mathcal{Z}$ *into* Δ^+ *and* $\mathcal{H}$ *is a t-norm, is called a Menger space if the following statements are satisfied for all* p,q,r *in* $\mathcal{Z}$ *and* v,w *in* $[0,\infty)$

1. $\mathcal{F}_{p,p}=\varepsilon_0$;
2. $\mathcal{F}_{p,q}\neq\varepsilon_0$ *if* $p\neq q$;
3. $\mathcal{F}_{p,q}=\mathcal{F}_{q,p}$;
4. $\mathcal{F}_{p,q}(v+w)\geq\mathcal{H}(\mathcal{F}_{p,r}(v),\mathcal{F}_{r,q}(w))$.

We should note that if $(\mathcal{Z},\mathcal{F},\mathcal{H})$ is a Menger space where $\sup_{a<1}\mathcal{H}(a,a)=1$, then $(\mathcal{Z},\mathcal{F},\mathcal{H})$ is a Hausdorff topological space [15] in the topology induced by the familly of (ε,λ)-neighborhoods defined as

$$\mathcal{N}=\left\{\mathcal{N}_p(\varepsilon,\lambda):p\in\mathcal{Z},\varepsilon>0\ \ and\ \ \lambda>0\right\}.$$

Where

$$\mathcal{N}_p(\varepsilon,\lambda)=\left\{q\in\mathcal{Z}:\mathcal{F}_{p,q}(\varepsilon)>1-\lambda\right\}.$$

Definition 2.5: *Let* $(\mathcal{Z},\mathcal{F},\mathcal{H})$ *be a Menger space. A sequence* $\{x_n\}$ *in* $\mathcal{Z}$ *is said to be:*

1. Convergent to $x\in\mathcal{Z}$ if for any given $\varepsilon>0$ and $\lambda>0$ there exist a positive integer $N=N(\varepsilon,\lambda)$ such that $\mathcal{F}_{x_n,x}(\lambda)>1-\varepsilon$ whenever $n\geq N$.
2. A Cauchy sequence if for any $\varepsilon>0$ and $\lambda>0$ there exist a positive integer $N=N(\varepsilon,\lambda)$ such that $\mathcal{F}_{x_n,x_m}(\lambda)>1-\varepsilon$ whenever $n,m\geq N$.

A Menger space $(\mathcal{Z},\mathcal{F},\mathcal{H})$ is said to be complete if each Cauchy sequence in $\mathcal{Z}$ is convergent to some point in $\mathcal{Z}$.

Definition 2.6: [16] *The triple* $(\mathcal{Z},\mathcal{M},\mathcal{H})$*, where* $\mathcal{Z}$ *is an arbitrary nonempty set,* $\mathcal{H}$ *is a t-norm and* $\mathcal{M}$ *is a fuzzy set on* $\mathcal{Z}\times\mathcal{Z}\times(0,+\infty)$*, is called a fuzzy metric space if the following statements are verified:*

1. $\mathcal{M}(p,q,v)>0$;
2. $\mathcal{M}(p,q,v)=1$ if and only if $p=q$;
3. $\mathcal{M}(p,q,v)=\mathcal{M}(q,p,v)$;
4. $\mathcal{M}(p,q,v+w)\geq\mathcal{H}\big(\mathcal{M}(p,q,v),\mathcal{M}(q,r,w)\big)$;
5. $\mathcal{M}(p,q,.):(0,+\infty)\to[0,1]$ is continuous for all $p,q,r\in\mathcal{Z}$ and $v,w>0$.

Definition 2.7: *Let* $(\mathcal{Z},\mathcal{F},\mathcal{H})$ *be a Menger space,* A *and* B *are two self maps of* $\mathcal{Z}$. *The pair* (A, B) *is said:*

1. Compatible if $\mathcal{F}_{\mathrm{AB}x_n,\mathrm{BA}x_n}\underset{n\to\infty}{\to}\varepsilon_0$ whenever the sequence $\{x_n\}$ satisfy $\lim \mathrm{A}x_n=\lim \mathrm{B}x_n=r$ for some $r\in\mathcal{Z}$.
2. Are weakly compatible if they commute at their coincidence points, this means, if $\mathrm{AB}x=\mathrm{BA}x$ whenever $\mathrm{A}x=\mathrm{B}x$.

Note that if the pair (A, B) is compatible, then it's trivially weakly compatible.

3. Main Results

Let Φ be the family of all functions $\omega:[0,+\infty)\to[0,+\infty)$ satisfying

$$\omega(t)<t\quad and\quad \lim_{n\to\infty}\omega^n(t)=0\quad for\ each\ t>0.$$

Before stating our main result, we will need the following lemma.

Lemma 3.1: *Let* $(\mathcal{Z},\mathcal{F},\mathcal{H})$ *be a Menger space,* $x,y\in\mathcal{Z}$ *and* $\omega\in\Phi$*, if*

$$\mathcal{F}_{x,y}(\omega(t))\geq\mathcal{F}_{x,y}(t)\quad\forall t>0, \tag{1}$$

then $x=y$.

Proof. From (1), for every $t>0$ we obtain that

$$\begin{aligned}\mathcal{F}_{x,y}(\omega^{n}(t))&\geq\mathcal{F}_{x,y}(\omega^{n-1}(t))\\&\geq\mathcal{F}_{x,y}(\omega^{n-2}(t))\\&\ \vdots\\&\geq\mathcal{F}_{x,y}(t).\end{aligned}$$

that is

$$\mathcal{F}_{x,y}(\omega^{n}(t))\geq\mathcal{F}_{x,y}(t). \tag{2}$$

Then it's obvious that $\mathcal{F}_{x,y}(t)=1$ for each $t>0$. Indeed, if there exists some u such that $\mathcal{F}_{x,y}(u)<1$, then since that $\lim_{u\to\infty}\mathcal{F}_{x,y}(u)=1$, there exists $v>u$ such that

$$\mathcal{F}_{x,y}(v)>\mathcal{F}_{x,y}(u). \tag{3}$$

And from that $\omega\in\Phi$, there exists $n\in\mathbb{N}$ satisfy $\omega^{n}(v)<u$.

By (2), (3) and the monotony of $\mathcal{F}_{x,y}(.)$ we have

$$\mathcal{F}_{x,y}(v)\leq\mathcal{F}_{x,y}(\omega^{n}(v))\leq\mathcal{F}_{x,y}(u)<\mathcal{F}_{x,y}(v),$$

which is not true.

Therefore, $\mathcal{F}_{x,y}(t)=1$ for each $t>0$.

Which implied that $x=y$.

Now we present the concept of (A,B)-ω-probabilistic contraction.

Definition 3.2: *Let* $(\mathcal{Z},\mathcal{F},\mathcal{H})$ *be a Menger space,* S *a nonempty subset of* $\mathcal{Z}$*,* A*,* B *and* h *are three self maps of* S*. If there exists* $\omega\in\Phi$ *such that the following inequality holds*

$$\mathcal{F}_{\mathrm{h}x,\mathrm{h}y}(\omega(t))\geq\mathcal{F}_{\mathrm{A}x,\mathrm{B}y}(t)\quad\textit{for all } x,y\in S, t>0,$$

then h is called (A,B)-ω-probabilistic contraction.

When $\mathrm{A}=\mathrm{B}=id_{Z}$ then h is called ω-probabilistic contraction [1].

Theorem 3.3: *Let* $(\mathcal{Z},\mathcal{F},\mathrm{H})$ *be a complete Menger space with* $\mathcal{H}$ *is of H-type and* $Ran\mathcal{F}\subset D^{+}$*. let* S *be a nonempty subset of* $\mathcal{Z}$*,* A,B *and* h *are three self maps of* S *which the following statements are verified:*

1. $\overline{h(S)}\subset\mathrm{A}(S)\cap\mathrm{B}(S)$;
2. h is (A,B)-ω-probabilistic contraction.

Then, there exists $p,q,r\in S$ such that $\mathrm{A}p=\mathrm{h}p=r=\mathrm{h}q=\mathrm{B}q$.

Furthermore, if (h,A) and (h,B) are weakly compatible, then r is the unique common fixed point of A, B and h.

Proof. Let $x_0\in S$, since $\overline{h(S)}\subset\mathrm{A}(S)\cap\mathrm{B}(S)$, then $\mathrm{h}(x_0)\in\mathrm{A}(S)\cap\mathrm{B}(S)$.

So there exists $x_1\in S$ such that $\mathrm{h}x_0=\mathrm{A}x_1$.

And since $\mathrm{h}x_1\subset\mathrm{A}(S)\cap\mathrm{B}(S)$, there exists also $x_2\in S$ such that $\mathrm{h}x_1=\mathrm{B}x_2$.

Then, by induction, we can construct a sequence $\{x_n\} \subset S$ satisfied

$$\mathrm{h}x_{2n} = \mathrm{A}x_{2n+1} \quad and \quad \mathrm{h}x_{2n+1} = \mathrm{B}x_{2n+2} \quad \forall n \in \mathbb{N}.$$

From that h is (A,B)-ω-probablistic contraction, then there exists $\omega \in \Phi$ such that

$$\begin{aligned} F_{\mathrm{h}x_{2n},\mathrm{h}x_{2n+1}}(\omega(t)) &\geq F_{\mathrm{A}x_{2n+1},\mathrm{B}x_{2n}}(t) \quad \forall t > 0 \\ &\geq F_{\mathrm{h}x_{2n},\mathrm{h}x_{2n-1}}(t). \end{aligned}$$

And

$$\begin{aligned} F_{\mathrm{h}x_{2n+1},\mathrm{h}x_{2n+2}}(\omega(t)) &\geq F_{\mathrm{A}x_{2n+1},\mathrm{B}x_{2n+2}}(t) \quad \forall t > 0 \\ &\geq F_{\mathrm{h}x_{2n},\mathrm{h}x_{2n+1}}(t). \end{aligned}$$

Then

$$F_{\mathrm{h}x_{n+1},\mathrm{h}x_n}(\omega(t)) \geq F_{\mathrm{h}x_n,\mathrm{h}x_{n-1}}(t) \quad \forall t > 0.$$

By induction we get

$$F_{\mathrm{h}x_{n+1},\mathrm{h}x_n}(\omega^n(t)) \geq F_{\mathrm{h}x_0,\mathrm{h}x_1}(t) \quad \forall t > 0.$$

Before we shaw that $\{\mathrm{h}x_n\}$ is a Cauchy sequence, we prove firstly that for all $\delta > 0$ and $0 < \varepsilon < 1$, there exists $n_0 \in \mathbb{N}$ satisfy

$$F_{\mathrm{h}x_{n+1},\mathrm{h}x_n}(\delta) > 1 - \varepsilon \quad \forall n \geq n_0.$$

Since $\lim_{t\to\infty} F_{\mathrm{h}x_0,\mathrm{h}x_1}(t) = 1$, there exist a t_1 such that

$$F_{\mathrm{h}x_0,hx_1}(t_1) > 1 - \varepsilon.$$

As $\omega \in \Phi$ and $\delta > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$\omega^n(t_1) < \delta \quad \forall\, n \geq n_1.$$

So, for $n \geq n_1$ we have

$$\begin{aligned} F_{\mathrm{h}x_{n+1},\mathrm{h}x_n}(\delta) &\geq F_{\mathrm{h}x_{n+1},\mathrm{h}x_n}(\omega^n(t_1)) \\ &\geq F_{\mathrm{h}x_0,\mathrm{h}x_1}(t_1) \\ &\geq 1 - \varepsilon. \end{aligned}$$

Hence

$$\lim_{n\to\infty} F_{\mathrm{h}x_n,\mathrm{h}x_{n+1}}(t) = 1.$$

Now, we prove that $\{\mathrm{h}x_n\}$ is a Cauchy sequence.

Since that for each $\delta > 0$, we have

$$F_{\mathrm{h}x_{n+1},\mathrm{h}x_n}(\delta) \geq F_{\mathrm{h}x_{n+1},\mathrm{h}x_n}(\delta - \omega(\delta)).$$

Then, we have

$$\begin{aligned} F_{\mathrm{h}x_{n+1},\mathrm{h}x_n}(\delta - \omega(\delta)) &\geq \mathcal{H}(F_{\mathrm{h}x_{n+1},\mathrm{h}x_n}(\delta - \omega(\delta)), F_{\mathrm{h}x_{n+1},\mathrm{h}x_n}(\delta - \omega(\delta))) \\ &\geq \mathcal{H}^1(F_{\mathrm{h}x_{n+1},\mathrm{h}x_n}(\delta - \omega(\delta))). \end{aligned}$$

Next by induction we get

$$F_{hx_{n+m},hx_n}(\delta) \geq \mathcal{H}^m(F_{hx_{n+1},hx_n}(\delta - \omega(\delta))) \quad \forall m \geq 1 \tag{4}$$

Let $\varepsilon \in (0,1)$. Since $\mathcal{H}$ is a t-norm of H-type, then there exist $\lambda \in (0,1)$ such that

$$t > 1 - \lambda \Rightarrow \mathcal{H}^p(t) > 1 - \varepsilon \quad \forall p \geq 1.$$

From that $\lim_{n\to\infty} F_{hx_n,hx_{n+1}}(t) = 1$, we get for $\delta - \omega(\delta) > 0$ that there exist $n_2 \in \mathbb{N}$ such that

$$F_{hx_{n+1},hx_n}(\delta - \omega(\delta)) > 1 - \lambda \quad \forall n > n_2.$$

And by (4) we conclude that

$$F_{hx_{n+m},hx_n}(\delta) > 1 - \varepsilon \quad \forall m \in \mathbb{N}, n \geq n_2.$$

Thus proved that $\{hx_n\}$ is a Cauchy sequence.

Since $\overline{h(S)}$ is complete, there exists $r \in S$ such that $\lim_{n\to\infty} hx_n = r \in \overline{h(S)}$.

While $\overline{h(S)} \subset A(S) \cap B(S)$, then there exists $p, q \in S$ such that $Ap = r = Bq$.

Now we shaw that $hp = hq = r$.

Let $t > 0$, we have

$$\begin{aligned} F_{hx_{2n+2},hp}(\omega(t)) &\geq F_{Bx_{2n+2},Ap}(t) \\ &\geq F_{hx_{2n+1},r}(t). \end{aligned}$$

Letting $n \to +\infty$, we obtain $F_{r,hp}(t) = 1$ for all $t > 0$, then we get $hp = r$.

Alike, we prove that $hq = r$.

Additionally, we suppose that (A,h) and (B,h) are weakly compatible and we shall prove that r is the unique common fixed point of h, A and B.

Since $hp = Ap$ and $hq = Bq$, we obtain $hAp = Ahp$ and $hBq = Bhq$. Then $hr = Ar = Br$.

From h is (A,B)-ω-contraction we get

$$F_{hp,hr}(\omega(t)) = F_{r,hr}(\omega(t)) \geq F_{Ap,Br}(t) = F_{r,hr}(t) \quad \forall\, t > 0.$$

Then, from Lemma 3.1 we conclude that $hr = r$.

Therefore r is a common fixed point of h, A and B.

Finally, we show the uniqueness of r.

If there exists another common fixed point y of h, A and B, then we have

$$F_{r,y}(\omega(t)) = F_{hr,hy}(\omega(t)) \geq F_{Ar,By}(t) = F_{r,y} \quad \forall t > 0.$$

Which implies from Lemma 3.1 that $r = y$. That completes the proof.

Remark 3.4: *Note that the condition of $RanF \subset D^+$ is a necessary condition to assure the uniqueness of the common fixed points if they exists as shown the next example.*

Example 3.5: *We put $\mathcal{Z} = \{p, q, r, w\}$, $F_{xy}(t) = \varepsilon_\infty(t)$ for each $x, y \in \mathcal{Z}$, $x \neq y$ and $t > 0$.*

Then it's easy to see that $(\mathcal{Z}, F, \mathcal{H}_M)$ is a complete Menger space under a t-norm of H-type.

Let $h, A, B : \mathcal{Z} \to \mathcal{Z}$ be three mappings which are defined by $hp = p, hq = q, hr = w, hw = r$ and $A = B = Id_Z$, we have, $\overline{h(\mathcal{Z})} \subset A(\mathcal{Z}) \cap B(\mathcal{Z})$ and (h,A) and (h,B) are weakly compatible. In addition

$$F_{hxhy}(\omega(t)) = F_{AxBy}(t),$$

for each $t > 0, x, y \in \mathcal{Z}, \omega \in \Phi$. So h is (A, B)-$\omega$-contraction.

It's should that h, A and B admits a unique common fixed point which is not the case, cause $F_{xy} \notin D^+$.

Remark 3.6: *We should also note also that the statement of $\mathcal{H}$ is of H-type is an essential hypothesis to prove the existence of common fixed point. We confirm this by the following example.*

Example 3.7: [17] *Let $\mathcal{L}$ be a distance distribution function which is defined by*

$$\mathcal{L}(t) = \begin{cases} 1 - \dfrac{1}{a} & \text{if} \quad 2^a < t < 2^{a+1},\ a > 1, \\ 0 & \text{if} \quad t \le 4. \end{cases}$$

We take $\mathcal{Z} = \{1, 2, \cdots\}$ and we define $F : \mathcal{Z} \times \mathcal{Z} \to D^+$ by

$$F_{a,a+b}(t) = \begin{cases} 0 & \text{if} \quad t = 0, \\ \mathcal{H}_L^{b+1}\left(\mathcal{L}(2^a t), \mathcal{L}(2^{a+1} t), \cdots, \mathcal{L}(2^{a+b} t)\right) & \text{if} \quad t > 0. \end{cases}$$

Then we obtain that $(\mathcal{Z}, F, \mathcal{H}_L)$ is a complet Menger space and the mapping $\mathrm{h}a = a + 1$ is (A, B)-ω-contractive with $\omega(t) = \frac{1}{2} t$ and $\mathrm{A} = \mathrm{B} = Id_{\mathcal{Z}}$. It clearly that $\overline{\mathrm{h}(\mathcal{Z})} \subset \mathrm{A}(\mathcal{Z}) \cap \mathrm{B}(\mathcal{Z})$ and (h, A) and (h, B) are weakly compatible. But h, A and B have no common fixed point, since that $\mathcal{H}_L$ is not of H-type.

By taking $\mathrm{A} = \mathrm{B} = id_S$ we obtain as a direct consequence of Theorem 3.3 the following result.

Corollary 3.8: [10] *Let $(\mathcal{Z}, F, \mathcal{H})$ be a complete Menger space with $\mathcal{H}$ is a continuous t-norm $\mathcal{H}$ of H-type and $RanF \subset D^+$. If* h *is a ω-probabilistic contraction mapping on $\mathcal{Z}$, then* h *has a unique fixed point r. As well, the sequence $(\mathrm{h}^n x)$ converges to r.*

The fuzzy version of Theorem 3.3 will be as the following.

Corollary 3.9: *Let $(\mathcal{Z}, \mathcal{M}, \mathcal{H})$ be a complete fuzzy metric space with $\mathcal{H}$ is of H-type and $\lim_{t \to \infty} \mathcal{M}(x, y, t) = 1$ for all $x, y \in \mathcal{Z}$. let S be a nonempty subset of $\mathcal{Z}$,* A, B *and* h *are three self maps of S which the following statements are verified:*

1. $\overline{\mathrm{h}(S)} \subset \mathrm{A}(S) \cap \mathrm{B}(S)$;
2. There's $\omega \in \Phi$ such that

$$\mathcal{M}(\mathrm{h}x, \mathrm{h}y, \omega(t)) \ge \mathcal{M}(\mathrm{A}x, \mathrm{B}y, t) \quad \textit{for all } x, y \in S \textit{ and } t \ge 0.$$

Then, there exists $p, q, r \in S$ such that $\mathrm{A}p = \mathrm{h}p = r = \mathrm{h}q = \mathrm{B}q$.

Furthermore, if (h, A) and (h, B) are weakly compatible, then r is the unique common fixed point of A, B and h.

Remark 3.10: *Since that the condition of "$F_{p,q}(0) = 0$" has not been used in the proof of Theorem 3.3, then the previous corollary remains true.*

Theorem 3.11: *Let $(\mathcal{Z}, F, \mathcal{H})$ be a complete Menger space with $\mathcal{H}$ is of H-type and $RanF \subset D^+$. let S be a nonempty subset of $\mathcal{Z}$,* A, B *be two self maps of S. Suppose $(\mathrm{h}_i)_{i \in I}$ is a family of self maps on S that verified the following statements:*

1. For $i \in I$, $\overline{\mathrm{h}_i(S)} \subset \mathrm{A}(S) \cap \mathrm{B}(S)$;
2. There's $\omega \in \Phi$ such that:

$$F_{\mathrm{h}_i x, \mathrm{h}_j y}(\omega(t)) \ge F_{\mathrm{A}x, \mathrm{B}y}(t) \quad \forall t \ge 0 \quad \forall i, j \in I \quad \forall x, y \in S;$$

3. For $i \in I$, $(\mathrm{h}_i, \mathrm{A})$ and $(\mathrm{h}_i, \mathrm{B})$ are weakly compatibles.

Then $(\mathrm{h}_i)_{i\in I}$, A and B admits a unique common fixed point.

Proof. By Theorem 3.3, we obtain that for each $i \in I$, h_i, A and B have a common fixed point r_i.

To complete the demonstration, we need to prove that: $r_i = r_j \ \ \forall i \neq j \in I$.

Let $t > 0$ and $i \neq j \in I$. So

$$\begin{aligned} \mathcal{F}_{r_i,r_j}(\omega(t)) &= \mathcal{F}_{\mathrm{h}_i r_i, \mathrm{h}_j r_j}(\omega(t)) \\ &\geq \mathcal{F}_{\mathrm{A}r_i,\mathrm{B}r_j}(t) \\ &= \mathcal{F}_{r_i,r_j}(t). \end{aligned}$$

From Lemma 3.1 we concluded that $r_i = r_j$, which is complete the proof.

Corollary 3.12: *Let $(\mathcal{Z},\mathcal{F},\mathcal{H})$ be a complete Menger space with $\mathcal{H}$ is of H-type and $Ran\mathcal{F} \subset D^+$. let S be a nonempty subset of $\mathcal{Z}$, A, B be two self maps of S. Suppose $(\mathrm{h}_i)_{i\in I}$ is a family of self maps on S that verified the following statements:*

1. For $i \in I$, $\overline{\mathrm{h}_i(S)} \subset \mathrm{A}(S) \cap \mathrm{B}(S)$;
2. There's $k \in (0,1)$ such that

$$\mathcal{F}_{\mathrm{h}_i x, \mathrm{h}_j y}(kt) \geq \mathcal{F}_{\mathrm{A}x,\mathrm{B}y}(t) \ \ \forall t \geq 0 \ \ \forall i,j \in I \ \ \forall x,y \in S;$$

3. For $i \in I$, $(\mathrm{h}_i, \mathrm{A})$ and $(\mathrm{h}_i, \mathrm{B})$ are weakly compatibles.

Then $(\mathrm{h}_i)_{i\in I}$, A and B admits a unique common fixed point.

4. Relatives Results in Ordinary Metric Spaces

As direct consequences of the above results, we obtain the corresponding fixed point theorem in usual metric spaces.

Theorem 4.1: *Let $(\mathcal{Z},\rho)$ be a complete metric space, S a nonempty set of $\mathcal{Z}$ and we suppose that* h*, A and B are self-maps of S such that $\overline{h(S)} \subset \mathrm{A}(S) \cap \mathrm{B}(S)$. If there exist a non-decreasing $\omega \in \Phi$ with $\omega(t) > 0$ for all $t > 0$ satisfy:*

$$\rho(\mathrm{h}x,\mathrm{h}y) \leq \omega\big(\rho(\mathrm{A}x,\mathrm{B}y)\big) \ \ \forall x,y \in \mathcal{Z}. \tag{5}$$

Then, there exists $p,q,r \in S$ such that $\mathrm{A}p = \mathrm{h}p = r = \mathrm{h}q = \mathrm{B}q$.

Furthermore, if (h,A) and (h,B) are weakly compatible, then r is the unique common fixed point of A, B and h.

Proof. We define $\mathcal{F} : \mathcal{Z} \times \mathcal{Z} \to D^+$ by

$$\mathcal{F}_{x,y}(t) = \begin{cases} 0 & if \quad \rho(x,y) \geq t, \\ 1 & if \quad \rho(x,y) < t. \end{cases}$$

Then $(\mathcal{Z},\mathcal{F},\mathcal{H}_M)$ is a complete Menger space (see [1] Lemma 3.5).

It suffices to show that the condition (7) assumed that h is (A,B)-ω-probabilistic contraction.

For an arbitrary $t > 0$, if $\mathcal{F}_{\mathrm{A}x,\mathrm{B}y}(t) = 0$, then it is clear that h is (A,B)-ω-probabilistic.

If $\mathcal{F}_{\mathrm{A}x,\mathrm{B}y}(t) = 1$, then we have $\rho(\mathrm{A}x,\mathrm{B}y) \leq t$, and from that ω is non-decreasing function, it implies that

$$\rho(\mathrm{h}x,\mathrm{h}y) \leq \omega\big(\rho(\mathrm{A}x,\mathrm{B}y)\big) \leq \omega(t).$$

Hence

$$F_{\mathrm{hx,hy}}(\omega(t)) = 1 = F_{\mathrm{Ax,By}}(t).$$

So h is (A,B)-ω-probabilistic contraction. Hence, the existence and the uniqueness follows from Theorem 3.3.

Similarly, we obtain from Corollary 4.1 the next result.

Corollary 4.2: *Let* $(\mathcal{Z},\rho)$ *be a complete metric space,* S *a nonempty set of* $\mathcal{Z}$ *and we suppose that* A *and* B *are self-maps of* S *. Let* $(\mathrm{h}_i)_{i\in I}$ *be a family of self-maps on* S *such that the following statements are verified:*

1. For $i \in I$, $\overline{\mathrm{h}_i(S)} \subset \mathrm{A}(S) \cap \mathrm{B}(S)$;
2. There's a non-decreasing $\omega \in \Phi$ with $\omega(t) > 0$ for all $t > 0$ satisfy:

$$\rho(\mathrm{h}x,\mathrm{h}y) \le \omega\big(\rho(\mathrm{A}x,\mathrm{B}y)\big) \quad \forall x, y \in \mathcal{Z}. \tag{6}$$

3. For $i \in I$, $(\mathrm{h}_i,\mathrm{A})$ and $(\mathrm{h}_i,\mathrm{B})$ are weakly compatibles.

Then $(\mathrm{h}_i)_{i\in I}$, A and B admits a unique common fixed point.

We point out that the above corollary allows us to extend the Jungck's common fixed point [8].

Corollary 4.3: *Let* $(\mathcal{Z},\rho)$ *be a complete metric space,* S *a nonempty set of* $\mathcal{Z}$ *and we suppose that* A *and* B *are self-maps of* S *. Let* $(\mathrm{h}_i)_{i\in I}$ *be a family of self-maps on* S *such that the following statements are verified:*

1. For $i \in I$, $\overline{\mathrm{h}_i(S)} \subset \mathrm{A}(S) \cap \mathrm{B}(S)$;
2. There's $k \in (0,1)$ such that

$$\rho(\mathrm{h}x,\mathrm{h}y) \le k\big(\rho(\mathrm{A}x,\mathrm{B}y)\big) \quad \forall x, y \in \mathcal{Z}. \tag{7}$$

3. For $i \in I$, $(\mathrm{h}_i,\mathrm{A})$ and $(\mathrm{h}_i,\mathrm{B})$ are compatibles.

Then $(\mathrm{h}_i)_{i\in I}$, A and B admits a unique common fixed point.

5. Conclusion

This paper demonstrates the existence and uniqueness of common fixed point for tree mappings under some contraction hypothesis, we have examined and extended this property of a family of mappings. In the objectif to illustrate our results we have given somme relatives results in ordinary metric space.

References

[1] Fang, J. Z., *On φ-contractions in probabilistic and fuzzy metric spaces*, Fuzzy Sets and Systems 267, (2015), 86–99.

[2] Fang, J. Z., *Common fixed point theorems of compatible and weakly compatible maps in Menger spaces*, Nonlinear Analysis 71, (2009), 1833–1843.

[3] Xiao, J. Z., Zhu, Z. H., Cao, Y. F., *Common coupled fixed point results for probabilistic φ-contractions in Menger spaces*, Nonlinear Anal. 74, (2011), 4589–4600.

[4] Kumar, S., Pant, B. D., *Some common fixed point theorems for mappings satisfying a new contraction condition in Menger spaces*, Varahmihir J. Math. Sci. 5, (2005), 227–234.

[5] Mishra, S.N., Common fixed points of compatible mappings in PM-spaces, Math. Japonica 36, (1991), 283–289.

[6] Singh, S. L., Mishra, S. N., Pant, B. D., *General fixed point theorems in probabilistic metric and uniform space*, Indian J. Math. 29, (1987), 9–21.

[7] Singh, S. L. Pant, B. D., Common fixed point theorems for commuting mappings in probabilistic metric spaces, Honam Math. J. 5, (1983), 139–150.

[8] Jungck, G., *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci. 9, (1986), 771–779.

[9] Singh, B., Jain, S., *A fixed point theorem in Menger space through weak compatibility*, J. Math. Anal. Appl. 301, (2005), 439–448.

[10] Jachymski, J., *On probabilistic ϕ-contractions on Menger spaces*, Nonlinear Analysis 73, (2010), 2199–2203.

[11] Banach, S., *Sur les operations dans les ensembles abstraites et leurs applications*, Fund. Math. 3, (1922), 133–181.

[12] Radu, V., *Some fixed point theorems in probabilistic metric spaces, Stability Problems for Stochastic Models (Verna, 1985)*, Lecture Notes in Math. 1233, (1987), 125–133.

[13] Sehgal, V. M., Bharucha-Reid, A. T., *Fixed points of contraction mappings on probabilistic metric space*, Math. Systems Theory 6, (1972), 97–102.

[14] Zhang, S., Zhu, Q., Song, Y., *Alternating picard itearates for hybird Boyd-Wong contractions*, Int. Journal of math. Analysis. 2, (2008), 563–568.

[15] Hadzic, O., Pap, E., *Fixed Point Theory in Probabilistic Metric Spaces*, Kluwer Academic Publishers, Dordrecht, (2001).

[16] George, A., Veeramani, P., *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, 64, (1994), 395–399.

[17] Sherwood, H., *Complete probabilistic metric spaces*, Z. Wahrscheinlichkeits theorie und Verw. Gebiete 20 (1971/72), 117–128.