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KKM implies Hahn-Banach

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Abstract

Our title means that the Knaster-Kuratowski-Mazurkiewicz theorem in 1929 implies the Hahn-Banach theorem. This theorem originated from Hahn in 1926 and Banach in 1929 is of basic importance in the analysis of problems concerning the existence of continuous linear functionals. Its consequences and applications cover hundreds of papers. For a long period, some authors studied the relation of results of the Hahn-Banach theorem and the Brouwer fixed point theorem in 1912 (equivalently, the KKM theorem in 1929). In the present article, we recall some history of such study, and show that the Hahn-Banach theorem can be derived from the KKM theorem and not conversely. Consequently, all consequences and applications of the Hahn-Banach theorem belong to a partial realm of the KKM theory.

Keywords: KKM theory; KKM theorem; Brouwer fixed point theorem; Hahn-Banach theorem.
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1. Introduction

In 1929, Knaster, Kuratowski, and Mazurkiewicz (simply, KKM) obtained an intersection theorem which is known to be equivalent to the Brouwer fixed point theorem in 1912 and the weak Sperner combinatorial lemma in 1928. The KKM theory is first named by ourselves in 1992 as the study of applications of generalizations or equivalents of the KKM theorem. Nowadays the realm of the theory is very broad; see Park [31].

Since 2006, we have introduced the new concepts of abstract convex spaces and (partial) KKM spaces which are adequate to establish the KKM theory. Recently, we have tried to enlarge the realm of the KKM theory of abstract convex spaces in 2017-2019; see [25]-[34].

Independently to the above progress, the Hahn-Banach theorem originated from Hahn [12] and Banach [1] is of basic importance in the analysis of problems concerning the existence of continuous linear functionals.

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Later there have been appeared a large number of articles on the Hahn-Banach theorem related to its generalization, variants, proofs, and applications; see MathSciNet and Google Scholar.

In 1974, Edwards [6] showed that a number of variational inequalities that had hitherto been proved by using the Brouwer fixed point theorem or the KKM theorem can in fact be deduced more simply from the Hahn-Banach theorem. Motivated by this fact, Simons in 1978-2008 [38]-[46] studied various results of the Hahn-Banach type or the Brouwer type (equivalently, the KKM type). He actually obtained the Hahn-Banach type proofs of some KKM type theorems.

In addition, some authors claim that even the Hahn-Banach theorem is of the Brouwer type or the KKM type; for example, see Hirano, Komiya, and Takahashi [13], and Granas and Lassonde [9], [10]. In fact, in [9] and [10], their authors added convex-valuedness to the classical KKM theorem and obtained a very large number of consequences in convex analysis and other fields.

Moreover, in our previous article [29], we showed that the distinction between the Hahn-Banach type and the KKM type is not strict, and that actually the two types are very close in a broad sense. Based on the article, we recollect some history of the study on the relation between the KKM theorem and the Hahn-Banach theorem.

In the present survey, by showing that KKM implies Hahn-Banach, we conclude that all consequences and applications of the Hahn-Banach theorem belong to a partial realm of the KKM theory.

Actually the present paper is a supplement to our previous [29].

2. The Hahn-Banach theorem

It begins with dominated extension theorems proved by H. Hahn in 1926 [12] and S. Banach in 1929 [1], respectively, by making use of the ideas of F. Riesz and E. Helly. Their theorems were later generalized to linear operators taking their values in normed spaces with the binary intersection property and ordered vector spaces.

Since then the Hahn-Banach theorem is of basic importance in the analysis of problems concerning the existence of continuous linear functionals. Its original proof in Banach [1], as well as the proofs given in most textbooks, relies on the axiom of choice.

We borrow the following version from Dunford and Schwartz [5]:

Theorem. (Hahn-Banach) *Let the real function p on the real linear space X satisfying*

$$p(x+y) \leq p(x) + p(y),\ p(\alpha x) = \alpha p(x);\quad \alpha \geq 0,\ x, y \in X.$$

Let f be a real linear functional on a subspace Y of X with

$$f(x) \leq p(x), \quad x \in Y.$$

Then there is a real linear functional F on X for which

$$F(x) = f(x), \quad x \in Y; \quad F(x) \leq p(x), \quad x \in X.$$

Here the proof is based on Zorn's lemma. Several consequences and comments on the theorem are given in Dunford-Schwartz ([5], Part I after p.62 and pp.85–88).

Early in 1938, Kakutani [15], [16] gave a proof of the Hahn-Banach extension theorem by using the Markov-Kakutani fixed point theorem, which follows from the Tychonoff fixed point theorem.

Rudin [36] stated that the plural is used for the theorem because the term "Hahn-Banach theorem" is customarily applied to several closely related results. Among them are the *dominated extension theorems* (in which no topology is involved), the *separation theorem*, and the *continuous extension theorem*. Moreover, his conclusion of the above theorem contains $-p(-x) \leq F(x) \leq p(x),\ x \in X$ instead of $F(x) \leq p(x),\ x \in X$, and his proof is also based on Zorn's lemma.

According to Simons [43], traditional proofs of the Hahn-Banach theorem were given by Kelley, Namioka et al. [17], Rudin [36], König [18], and Simons [37]. Such proofs used to depend on Axiom of Choice or Zorn's Lemma, but nowadays there have been appeared many proofs without using them. For more proofs, see also Buskes [4], Lassonde [19], and some others.

For some classical applications of the Hahn-Banach theorem, see the above mentioned references and Narici and Beckenstein [20].

3. KKM implies Hahn-Banach by Kakutani [16] and Fan [8]

Early in 1938, Kakutani [16] gave a proof of the Hahn-Banach extension theorem by using the Markov-Kakutani fixed point theorem, which follows from the Tychonoff fixed point theorem.

Theorem. (Markov 1936) *Let B be a non-vacuous, convex, bicompact subset of a locally convex linear topological space E, and let Γ be an abelian family of continuous affine transformation $\varphi(x)$ of B into itself; then there is a point $x \in B$ such that we have $\varphi(x) = x$ for any $\varphi \in \Gamma$.*

Kakutani [15] proved this theorem without assuming the local convexity and showed [16] that

$$\text{Markov-Kakutani theorem} \Longrightarrow \text{Hahn-Banach theorem.}$$

The following summarizes the steps that the KKM theorem implies the Hahn-Banach theorem:

KKM $\Longrightarrow$ KKMF (Fan's 1961 KKM lemma) — Fan in 1961 [8]
$\Longrightarrow$ Fan's section property — Fan in 1961 [8]
$\Longrightarrow$ Tychonoff's fixed point theorem — Fan in 1961 [8]
$\Longrightarrow$ Markov's fixed point theorem — Markov in 1934
$\Longrightarrow$ Hahn-Banach theorem — Kakutani in 1938 [16]
$\Longrightarrow$ Markov-Kakutani fixed point theorem — Takahashi in 1986 [47], Werner in 1992 [48]

Therefore, *the Hahn-Banach theorem is derived from the KKM theorem.* See also Park [29], [31]. All theorems appeared in the above diagram except Hahn-Banach can be seen in our previous work [21].

4. Equivalent forms of Hahn-Banach by Takahashi [47]

The results of Takahashi in 1986 [47] are as follows:

(1) A new proof, based on the Brouwer fixed point theorem, of a Fan's matching theorem in 1984.

(2) An extension to the noncompact case of a Fan-Browder type fixed point result due to H. Ben-El-Mechaiekh et al. in 1982 [3].

(3) A minimax theorem of Sion type, based on Fan's minimax inequality.

(4) A new proof of the Markov-Kakutani fixed point theorem, based on Fan's convex inequalities theorem in 1957 [7], which is equivalent to a minimax theorem of Fan in 1953 deduced from the Hahn-Banach theorem.

(5) A version of König's theorem, proved by using the same theorem of Fan, and then prove Simons' minimax theorem in 1981 [39].

At the end of Introduction of [47], the author added: Consequently, keeping in mind that the Hahn-Banach theorem is deduced from Brouwer's fixed point theorem (for example, see Hirano-Komiya-Takahashi [13]), we are able to compare the "strength" of results appearing in Simons [41] and the present paper [47].

Comments:

(1) and (3) are generalized in Park [22].

(2) is also generalized for abstract convex spaces by Park [22], [24], [26].

(3) Fan's convex inequality theorem is already deduced from the Brouwer fixed point theorem in Takahashi in 1976. Therefore, by (4), the Markov-Kakutani fixed-point theorem is of the Brouwer type, and implies the Hahn-Banach theorem by Kakutani [15], [16]. Consequently, the Hahn-Banach theorem is of the KKM type.

(4) shows that:

Hahn-Banach theorem $\Longrightarrow$ a minimax theorem of Fan in 1953

$\Longrightarrow$ Fan's convex inequalities theorem in 1957

$\Longrightarrow$ Markov-Kakutani theorem

$\Longrightarrow$ Hahn-Banach theorem (Kakutani [15]).

Therefore, already in Takahashi [47], the following result of Werner [48] was indicated:

Hahn-Banach theorem $\Longrightarrow$ Markov-Kakutani fixed point theorem.

Moreover, Hirano-Komiya-Takahashi in 1982 [13] used the Markov-Kakutani theorem to prove the following generalized form of the Hahn-Banach theorem.

Lemma 3. ([13]) *If p is sublinear on a linear space E and $x_0 \in E$, then there is an $f \in E^*$ such that $f(x) \leq p(x)$ for all $x \in E$ and $f(x_0) = p(x_0)$.*

From now on we introduce several articles related to the subject of the present one.

5. KKM implies Hahn-Banach by Granas and Lassonde [9]

According to Abstract, Granas and Lassonde [9] present a new elementary approach in the theory of minimax inequalities. The proof of the main result (called the geometric principle) uses only some simple properties of convex functions. The geometric principle (which is equivalent to the well-known lemma of Klee in 1951) is shown to have numerous applications in different areas of mathematics.

The geometric principle is as follows.

Theorem 1. ([9] Principe Géométrique) *Let E be a topological vector space, let $\emptyset \neq D \subset E$, and let G be a multimap from D to E such that*
(1) *$G(x)$ is a closed convex set for all $x \in D$, and*
(2) *G is a KKM map* ($\mathrm{co}\, A \subset G(A)$ *for all finite $A \subset D$);*
then the family $\{G(x) \mid x \in D\}$ has the finite intersection property.

According to MR1141361(93b:49009) by Jean-Pierre Crouzeix, the proof of this result is very simple and depends only on the geometric structure induced by convexity. Many applications are given (minimax, variational inequalities, maximal monotone operators, etc.). Unlike other approaches, the derivation of results is obtained without using arguments such as convexity, the separation theorem, and fixed point theory.

Actually, Theorem 1 of Granas and Lassonde [9] is a particular form of the well-known 1961 KKM lemma of Ky Fan [8]. Their proof is very simple and depends only on the geometric structure induced by convexity.
In [9], many applications of Theorem 1 to known results are systemically given on systems of inequalities, variational inequalities, minimax equalities, theorems of Markov-Kakutani, Mazur-Orlicz and Hahn-Banach, variational problems, maximal monotone operators, and others in convex analysis.

Note that they recalled the following theorem of Banach (version de base du Théoréme de Hahn-Banach):

Lemma 1. (Banach) *If $p : E \to \mathbf{R}$ is sublinear, then there exists $f \in E^*$ such that $f(x) \leq p(x)$ for all $x \in E$.*

Moreover, they deduced

Markov-Kakutani $\Longrightarrow$ Lemma 1 (Banach) $\Longrightarrow$ Lemma 2, and Lemma 2 (the same to Lemma 3 of Hirano-Komiya-Takahashi in 1982 [13]) generalizes the Hahn-Banach theorem.

They also showed that

KKM $\Longrightarrow$ Théorème 1 (Principe Géométrique)

$\Longleftrightarrow$ Corollaire 1.1

$\Longleftrightarrow$ Théorème 2.

This equivalencies may be generalized for abstract convex spaces as in Park [22].

Further, they deduced

Lemma 2 + Corollary 4.1 (Fan in 1957 [7] and others)

$\Longrightarrow$ Théorème 9 (Mazur-Orlicz)

$\Longrightarrow$ Corollaire 9.1

$\Longrightarrow$ Hahn-Banach, where

Corollary 9.1. ([9]) *Let $p : E \to \mathbb{R}$ be a sub-linear functional, $C \subset E$ a convex subset of a real vector space E, and $g : C \to \mathbb{R}$ a concave function such that $g(y) \le p(y)$ for all $y \in C$. Then there exists $f \in E^*$ such that $f(x) \le p(x)$ for all $x \in E$ and $f(y) \ge g(y)$ for every $y \in C$.*

Then their version of the Hahn-Banach theorem is an evident particular case of Corollary 9.1.

Later Ben-El-Mechaiekh [2] showed that

Markov-Kakutani $\Longleftrightarrow$ Geometric principle (the convex KKM principle in [2]).

Recall that

Markov-Kakutani $\Longrightarrow$ Lemme 2 — Hirano-Komiya-Takahashi in 1982 [13]

Geometric principle $\Longrightarrow$ Corollaire 4.1 — Granas and Lassonde [9].

Therefore, we have

KKM $\Longrightarrow$ Geometric principle (or Markov-Kakutani)

$\Longrightarrow$ Lemma 2 + Corollary 4.1 (Fan in 1957 [7] and others)

$\Longrightarrow$ Théorème 9 (Mazur-Orlicz)

$\Longrightarrow$ Corollaire 9.1

$\Longrightarrow$ Hahn-Banach.

This is another evidence of "KKM implies Hahn-Banach."

6. Park's KKM theory in 2010 [22]

The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open-valued" version.

In [22], we clearly derive a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces.

Consequently, this paper unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces.

In [22], statements (V), (VI), Theorem 4, (XVI), (XVII) are incorrectly stated and can be easily corrected. In this paper a large number of KKM type theorems are stated.

7. Article of Horvath in 2014 [14]

In 2014, Horvath [14] published a related article to [9]. It is shown in [14] that the Elementary KKM theorem is equivalent to Klee's theorem, the Elementary Alexandroff-Pasynkov theorem, the Elementary Ky Fan theorem and the Sion-von Neumann minimax theorem, as well as a few other classical results with an extra convexity assumption.

Recall that the KKM theorem, the Alexandroff-Pasynkov theorem, and the Ky Fan theorem are mutually equivalent; see [35]. Horvath showed that "elementary" versions of these theorems are equivalent to Klee's theorem and the Sion-von Neumann minimax theorem.

Theorem 2.1. (Klee's theorem, 1951) *If $A_0, \ldots, A_n$ is a family of closed convex sets such that, for all $j \in \{0, \ldots, n\}$, $\bigcap_{i \neq j} A_i \neq \emptyset$ and $\bigcup_{i=0}^n A_i$ is convex, then $\bigcap_{i=0}^n A_i \neq \emptyset$.*

Theorem 2.2. (Elementary KKM) *A family F_i, $i \in \{0, \ldots, n\}$, of closed and convex subsets of the n-dimensional simplex $\Delta_n = \mathrm{conv}\{e_i : i \in \{0, \ldots, n\}$ such that, for all nonempty subsets $J \subset \{0, \ldots, n\}$, $\Delta_J \subset \bigcup_{j \in J} F_j$, has a nonempty intersection.*

Theorem 2.3. (Elementary Alexandroff-Pasynkoff theorem, 1957) *Let $\{M_0, \ldots, M_n\}$ be a cover of Δ_n by closed convex sets. If, for all $i \in \{0, \ldots, n\}$, $\mathrm{conv}\{e_j : j \neq i\} \subset M_i$, then $\bigcap_{i=0}^n M_i \neq \emptyset$.*

The first chain of implications in [14] is as follows:

Klee $\Longrightarrow$ Elementary KKM

$\Longrightarrow$ Elementary Alexandroff–Pasynkoff

$\Longrightarrow$ Klee

Elementary Ky Fan's theorem. *Let X be a convex subset of a topological vector space and let $\Omega : X \multimap X$ be a multifunction with closed values such that*

(1) *for all $x \in X$, $x \in \Omega x$,*

(2) *for all $y \in X$, $X \setminus \Omega^{-1} y$ is convex.*

Then $x \mapsto \mathrm{conv}(\Omega x)$ has the finite intersection property.

Elementary Ky Fan's inequality. *Let X be a compact convex set and let $f : X \times X \to \mathbb{R}$ be a function such that*

(1) *for all $x \in X$, $f(x, x) \leq 0$,*

(2) *for all $x \in X$, $y \mapsto f(x, y)$ is l.s.c. and quasi-convex,*

(3) *for all $y \in X$, $x \mapsto f(x, y)$ is quasi-concave.*

Then, there exists $y_0 \in X$ such that, for all $x \in X$, $f(x, y_0) \leq 0$.

The second chain of implications:

Elementary KKM theorem $\Longrightarrow$ Geometric KKM principle

$\Longrightarrow$ Elementary Ky Fan's theorem

$\Longrightarrow$ Elementary Ky Fan's inequality

$\Longrightarrow$ von Neumann's minimax theorem

$\Longrightarrow$ Klee's theorem

$\Longrightarrow$ Elementary KKM theorem

The third chain of implications:

von Neumann's minimax theorem

$\Longrightarrow$ Klee's theorem

$\Longrightarrow$ Sion's theorem

$\Longrightarrow$ von Neumann's minimax theorem.

8. Ben-El-Mechaiekh's article in 2015 [2]

Ben-El-Mechaiekh [2] stated : "A number of landmark existence theorems of nonlinear functional analysis follow in a simple and direct way from the basic separation of convex closed sets in finite dimension via elementary versions of the Knaster-Kuratowski-Mazurkiewicz principle — which we extend to arbitrary topological vector spaces — and a coincidence property for so-called von Neumann relations. The method avoids the use of deeper results of topological essence such as the Brouwer fixed point theorem or the Sperner's lemma and underlines the crucial role played by convexity. It turns out that the convex KKM principle is equivalent to the Hahn-Banach theorem, the Markov-Kakutani fixed point theorem, and the Sion-von Neumann minimax principle."

Note that the geometric principle of Granas and Lassonde [9] is called the "convexq KKM principle by Ben-El-Mechaiekh [2].

Actually Ben-El-Mechaiekh [2] derived the following:

Definition 8. A *von Neumann relation* is a subset A of a cartesian product $X \times Y$, where X and Y are subsets of topological vector spaces, satisfying:

(i) for every $x \in X$, the section $A(x)$ is convex and non-empty;

(ii) for every $y \in Y$, the section $A^{-1}(y)$ is open in X and $X \setminus A^{-1}(y)$ is convex.

Denote by $\mathcal{N}(X,Y)$ the class of von Neumann relations in $X \times Y$ and by $\mathcal{N}^{-1}(X,Y) := \{A : X \multimap Y \mid A^{-1} \in \mathcal{N}(Y,X)\}$.

Theorem 9. (Fixed Point for $\mathcal{N}$-maps) *Let E be a t.v.s., $\emptyset \neq Y \subset X \subset E$ with X convex, and let $A \in \mathcal{N}(X,Y)$. If there exist a compact subset K of X and a compact convex subset D of Y such that for every $x \in X \setminus K$, $A(x) \cap D \neq \emptyset$, then A has a fixed point, i.e., $(\hat{x}, \hat{x}) \in A$ for some $\hat{x} \in X$.*

Ben-El-Mechaiekh [2] noted that Theorem 9 is equivalent to the Klee intersection theorem and the convex KKM theorem (with certain compactness condition).

9. The Hahn-Banach type or the KKM type? [29]

In 2018, we published a survey article entitled as above. For a long period, some authors studied relations of results of the Hahn-Banach theorem or the Brouwer fixed point theorem (equivalently, the KKM theorem). In [29], we present large numbers of examples of the Hahn-Banach type or of the KKM type, and show that the Hahn-Banach theorem is the KKM type. Therefore, the distinction between these two types is not strict, and the two types are closely related in a broad sense.

In [29], it is incorrectly stated that Klee's intersection theorem in 1951 and Sion's minimax equality in 1958 are equivalent to the Brouwer fixe point theorem.

Since the Hahn-Banach theorem is equivalent to elementary KKM theorem by Ben-El-Mechaiekh [2], the Hahn-Banach type is of the KKM type in our previous work [29]. Note that this does not conversely hold since there are so many statements equivalent to the KKM principle. Since KKM implies Hahn-Banach, we conclude that all consequences and applications of the Hahn-Banach theorem belong to a partial realm of the KKM theory.

10. Generalization of the convex-valued KKM principle

The geometric principle of Granas and Lassonde [9] can be generalized to abstract convex spaces as follows.

Recall the following well-known related four conditions for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y\in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y\in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y\in D} \overline{G(y)} = \overline{\bigcap_{y\in D} G(y)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{y\in D} \overline{G(y)} = \bigcap_{y\in D} G(y)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

Note that Luc et al. showed that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d), and not conversely in each step.

The following is one of the most general KKM type theorems for abstract convex spaces in Park [23]:

Theorem C. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, D, Z)$, and $G : D \multimap Z$ a map such that*

(1) *$\overline{G}$ is a KKM map w.r.t. F; and*

(2) *there exists a nonempty compact subset K of Z such that either*

(i) *$K = Z$;*

(ii) *$\bigcap\{\overline{G(y)} \mid y \in M\} \subset K$ for some $M \in \langle D\rangle$; or*

(iii) *for each $N \in \langle D\rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, $\overline{F(L_N)}$ is compact, and*

$$\overline{F(L_N)} \cap \bigcap_{y\in D'} \overline{G(y)} \subset K.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if G is transfer closed-valued, then $\overline{F(E)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$; and*

(β) *if G is intersectionally closed-valued, then $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.*

Consider the case $E = Z$, $F = \mathrm{id}_E$, and the following instead of the conclusion:

(α) *if G is Γ-convex transfer closed-valued, then $K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$; and*

(β) *if G is Γ-convex intersectionally closed-valued, then $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.*

Then the conclusion generalizes the geometric principle of Granas and Lassonde [9], the elementary KKM theorem of Horvath [14], and the convex KKM principle of Ben-El-Mechaiekh [2].

11. Summary and Conclusion

This section is the revised version of Section 5 of [29].

In the history of the KKM theory, some authors tried to distinguish the Hahn-Banach type and the KKM type statements. The major claims on the relations between these two types can be summarized as follows:

1. Markov deduced his fixed point theorem in 1936 from Tychonoff's fixed point theorem in 1935.

2. Kakutani in 1938 showed that the Markov theorem implies the Hahn-Banach theorem, which follows from Tychonoff's theorem.

3. Fan in 1961 [8] deduced Tychonoff's theorem from the KKM theorem. Consequently, the Hahn-Banach theorem is of the KKM type by 1. and 2.

4. Takahashi in 1976 showed that the Markov-Kakutani fixed-point theorem is of the Brouwer type. Combining this with Kakutani [15], the Hahn-Banach theorem is of the Brouwer type.

5. Edwards in 1978 [6] showed that a number of variational inequalities proved by a fixed-point theorem or the KKM theorem can be deduced more simply from the Hahn-Banach theorem.

6. In 1978–2008, Simons published many articles related to the KKM type or the Hahn-Banach type; see the references in the end of this paper.

7. Simons in 1981 [39] first distinguishes the fixed point type and the Hahn-Banach type.

8. Hirano-Komiya-Takahashi in 1982 [13] noted that the Markov-Kakutani theorem implies a generalized form of the Hahn-Banach theorem.

9. Granas and Liu in 1983 [11]: The KKM theorem implies a theorem on systems of inequalities involving three families of functions.

10. Simons in 1985 [40]: The Hahn-Banach theorem implies the theorem of Granas and Liu [11] on systems of inequalities involving three families of functions.

11. Simons 1986 [41]: "We believe that §3 captures the essence of the fact that the KKM theorem and the Hahn-Banach theorem are saying different things".

12. Simons in 1989 [42] studied minimax and variational inequalities of the fixed point type or the Hahn-Banach type.

13. Granas and Lassonde in 1991 [9] : The elementary KKM theorem implies the Hahn-Banach theorem.

14. Werner 1992 [48] gave a proof of the Markov-Kakutani fixed point theorem via the Hahn-Banach theorem, and hence the two results are indeed equivalent. This was already known by Takahashi in 1986 [47].

15. Horvath in 2014 [14]: Elementary KKM theorem is equivalent to Klee's theorem, the elementary Alexandroff-Pasynkov theorem, the elementary Ky Fan theorem, and the Sion-von Neumann minimax theorem, as well as a few other classical results with an extra convexity assumption.

16. Ben-El-Mechaiekh in 2015 [2] : The convex KKM principle is equivalent to the Hahn-Banach theorem, the Markov-Kakutani fixed point theorem, and the Sion-von Neumann minimax principle.

17. Park in 2018 [29] : The Hahn-Banach type is a proper subclass of the KKM type.

Consequently, the Hahn-Banach type of Simons is the class of statements whose proofs are based on the Hahn-Banach theorem and shorter than the one using the Brouwer fixed point theorem or the KKM theorem. However, the Hahn-Banach theorem is of the KKM type and equivalent to a particular form of the KKM theorem. Therefore, the Hahn-Banach type is a proper subclass of the KKM type.

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