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Structural approach of infinite matrix using difference operator

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Abstract

The major structure of this present article is to establish and analyze the spaces involving the infinite matrices with the operator introduced by Kizmaz. β-duals will be constructed. BK spaces will be given its place for synthesis.

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Key words: Sequence spaces; β-duals; Kizmaz operator

1. Introduction

Considering the sequences of set of real or complex numbers as ω and where else ℓ_∞ c and c_0 be assumed as the linear spaces of the form bounded, convergent, and null sequences of complex terms [1–8]. H. Kizmaz [9], first introduced the difference sequence space and defined sequence spaces as

$$\ell_\infty(\Delta) = \{v = (v_i) : \Delta v \in \ell_\infty\}$$
$$c(\Delta) = \{v = (v_i) : \Delta v \in c\}$$

and

$$c_0(\Delta) = \{v = (v_i) : \Delta v \in c_0\}.$$

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where $\Delta v = \Delta v_i = (v_i - v_{i-1})$. As in [1], [10], [11], [12], [14]–[19], we define κ_B to be an infinite matrix B in the space κ as

$$\kappa_B = \{v = (v_i) \in \omega : Bv \in \kappa\}.$$

Definition 1.1: *Let the space Λ having linear topology be assumed as Kothe space if $P_m = \Lambda \to \mathbf{C}$ is continuous for $P_m(y) = y_m \forall y = (y_m) \in \Lambda$ for each $m \in \mathbb{N}$.*

Definition 1.2: *Let us consider a complete linear space as Frechet space. Also, the K-space Λ is known as FK-space if Λ will be a complete linear metric space. Therefore, A normed FK-space will be known as BK-space.*

Definition 1.3: *For non-negative entries $b_n k$, the matrix $B = (b_{nk})$ is said be be Kothe matrix if*

(i) For each $m \in \mathbf{N}$ if $\exists$ a natural number j such that $b_{mj} > 0$ and
(ii) $b_{mj} \le b_{mj+1}$ $\forall m, j \in \mathbf{N}$.

Definition 1.4: *Assuming Λ to be any one of the sequence spaces ℓ_∞, ℓ_1, c or c_0. Then, $\Lambda(\Delta)$ comprise of the sequences*

$$s = (s_r) \exists (s_r - s(r+1)) \in \Lambda$$

is difference sequences spaces. Also in [13], the difference sequence space Δ^m is given as follows:

$$\Delta^m \lambda = \{s = (s_r) \in \omega : \Delta^m s \in \lambda\},$$

where $\Delta^1 s = (s_r - s_{r+1})$ and $\Delta^m s = \Delta(\Delta^{m-1} s)$ for $m \in \{1,2,3,\cdots\}$.

Theorem 1.1: *As in* [9], *the space $\ell_\infty(\Delta)$ is considered as a Banach space under norm*

$$\| s \|_\Delta = | s_1 | + \| \Delta s \|_\infty .$$

Corollary 1.2: *Also the space $c(\Delta)$ and $c_0(\Delta)$ are assumed to be Banach spaces [4].*

2. Köthe matrix using Δ-approach

Throughout the article $\ell_\infty(\Delta_g, B)$, $\ell_1(\Delta_g, B)$, $\ell_p(\Delta_g, B)$, $c_0(\Delta_g, B)$ and $c_0(\Delta_g, B)$ are assumed as the class of bounded, summable, p-summable, null and convergent sequences spaces, where $B = (b_{nk})$ be a Köthe matrix [1].

Now for $g = (g_k)$ with $g_k \neq 0$ for all $k \in \mathbf{N}$, we define the following new spaces as:

$$\ell_1(\Delta_g, B) = \left\{ v = (v_n) \in \omega : \sum_{n=1}^{\infty} \left| \Delta_g v_n b_{nk} \right| < \infty \right\},$$

$$\ell_p(\Delta_g, B) = \left\{ v = (v_n) \in \omega : \left(\sum_{n=1}^{\infty} \left| \Delta_g v_n b_{nk} \right|^p \right)^{\frac{1}{p}} < \infty \right\},$$

$$c(\Delta_g, B) = \{ v = (v_n) \in \omega : \lim_{n \to \infty} (b_{nk} \Delta_g v_n - l) = 0 \},$$

$$c_0(\Delta_g, B) = \{ v = (v_n) \in \omega : \lim_{n \to \infty} \Delta_g v_n b_{nk} = 0 \},$$

$$\ell_\infty(\Delta_g, B) = \{ v = (v_n) \in \omega : \sup_n \left| \Delta_g v_n b_{nk} \right| < \infty \}.$$

Lemma 2.1: *Let the function* $\|\cdot\|_{(\Delta,B)}$ *be defined as*

$$\| x \|_{\Delta} = | g_1 v_1 b_{1k} | + \| \Delta_g v_n b_{nk} \|_{\infty}$$

be the norm on $\ell_{\infty}(\Delta_g, B)$, where B is assumed to be a Köthe Matrix.

Theorem 2.2: *The set* $\ell_{\infty}(\Delta_g, B)$ *is a vector space of* ω.

Proof: Suppose $v = (v_n)$ and $s = s_(n)$ are elements of $\ell_{\infty}(\Delta_g, B)$, such that

$$\| v \|_{\infty} = \sup_n | b_{nk} \Delta_g v | < \infty \; and \; \| s \|_{\infty} = | b_{nk} \Delta_g v | < \infty.$$

Now for the scalars a, b, we have

$$\begin{aligned} \| av + bs \| &= \sup_n | b_{nk} \Delta_g (a v_n + b s_n) | \\ &= \sup_n | b_{nk} (\Delta_g a v_n + \Delta_g b s_n) | \\ &= \sup_n | b_{nk} \Delta_g a v_n + b_{nk} \Delta_g b s_n | \\ &\le a \sup_n | b_{nk} \Delta_g v_n | + b \sup_n | b_{nk} \Delta_g s_n | \\ &< \infty. \end{aligned}$$

So, this shows that $av + bs \in \ell_{\infty}(\Delta_g, B)$. Consequently, $\ell_{\infty}(\Delta_g, B)$ is be a vector space of ω.

Theorem 2.3: *The sets* $\ell_1(\Delta_g, B)$, $\ell_p(\Delta_g, B)$, $c(\Delta_g, B)$ *and* $c_0(\Delta_g, B)$ *are linear spaces of* ω.

Proof: The result could be analogously and henceforth eliminated.

Theorem 2.4: *The sets* $\ell_1(\Delta_g, B)$, $\ell_p(\Delta_g, B)$, $c(\Delta_g, B)$, $c_0(\Delta_g, B)$ *and* $\ell_{\infty}(\Delta_g, B)$ *are normed spaces under*

$$\| v \|_{(\Delta,B)} = | g_1 v_1 b_{1k} | + \| \Delta_g v_n b_{nk} \|_{\infty} .$$

Proof: We only prove the result for $\ell_{\infty}(\Delta_g, B)$ and the rest could be demonstrated in the similar way. Also, the properties of norm for the function $\| v \|_{(\Delta,B)}$ are as follows:

$$(i) \quad \begin{aligned} \| v \|_{(\Delta,B)} &= | g_1 v_1 b_{1k} | + \| \Delta_g v_n b_{nk} \|_{\infty} \\ &= | g_1 v_1 b_{1k} | + \sup_n \{ | g_n v_n b_{nk} | \} \\ &\ge 0. \end{aligned}$$

$$(ii) \quad \| v \|_{(\Delta,B)} = 0 \Leftarrow (| g_1 v_1 b_{1k} | + \| \Delta_g v_n b_{nk} \|_{\infty}) = 0.$$

This implies that

$$\begin{aligned} & | g_1 v_1 b_{1k} | = 0 \; and \; \| \Delta_g v_n b_{nk} \|_{\infty} = 0 \\ \Rightarrow \quad & g_1 v_1 b_{1k} = 0 \; and \; \sup_n | \Delta_g v_n b_{nk} | = 0 \end{aligned}$$

But $g_1, b_{1k} \ne 0$, therefore, $v_1 = 0$ and $| \Delta_g v_n b_{nk} | = 0$.

Also, $g_k, b_{nk} \ne 0 \Rightarrow v_n - v_{n+1} = 0 \Rightarrow v_n = v_{n+1}$. Also, $v_1 = 0$, it follows that

$$v_n = 0 \forall n \in \mathbf{N}.$$

Therefore, we have

$$\| v \|_{(\Delta,B)} = 0 \Leftarrow v = 0.$$

$$\begin{aligned} (iii) \quad \| \alpha v \|_{(\Delta,B)} &= | \alpha g_1 v_1 b_{1k} | + \| \alpha \Delta_g v_n b_{nk} \|_\infty \\ &= | \alpha | \, | g_1 v_1 b_{1k} | + \sup_n \{ | \alpha g_n v_n b_{nk} | \} \\ &= | \alpha | \, | g_1 v_1 b_{1k} | + \sup_n \{ | \alpha | \, | g_n v_n b_{nk} | \} \\ &= | \alpha | \, | g_1 v_1 b_{1k} | + | \alpha | \sup_n \{ | g_n v_n b_{nk} | \} \\ &= | \alpha | [| g_1 v_1 b_{1k} | + \sup_n \{ | g_n v_n b_{nk} | \}] \\ &= | \alpha | \| v \|_{(\Delta,B)} . \end{aligned} \tag{1}$$

This shows that

$$\| \alpha v \|_{(\Delta,B)} = | \alpha | \| v \|_{(\Delta,B)} .$$

$$\begin{aligned} (iv) \quad \| u + v \|_{(\Delta,B)} &= | (u_1 g_1 + v_1 g_1) b_{1k} | + \| \Delta_g (u_n + v_n) b_{nk} \|_\infty \\ &= | g_1 u_1 b_{1k} + g_1 v_1 b_{1k} | + \sup_n \{ | \Delta_g (u_n + v_n) b_{nk} | \} \\ &\le | g_1 u_1 b_{1k} | + | g_1 v_1 b_{1k} | + \sup_n \{ | \Delta_g u_n b_{nk} | \} + \sup_n \{ | \Delta_g v_n b_{nk} | \} \\ &= [| g_1 u_1 b_{1k} | + \sup_n \{ | \Delta_g u_n b_{nk} | \}] + [| g_1 v_1 b_{1k} | + \sup_n \{ | \Delta_g v_n b_{nk} | \}] \end{aligned}$$

$$\Rightarrow \quad \| u + v \|_{(\Delta,B)} \le \| u \|_{(\Delta,B)} + \| u + v \|_{(\Delta,B)} .$$

Hence, $\ell_\infty(\Delta_g, B)$ is a normed linear space.

Theorem 2.5: *Assume $b_{n,k} \ge k \in \mathbf{R}^+$ for each $n,k \in \mathbf{N}$. Then, the spaces $\ell_\infty(\Delta_g, B)$, $\ell_1(\Delta_g, B)$, $\ell_p(\Delta_g, B)$, $c(\Delta_g, B)$ and $c_0(\Delta_g, B)$ are K-spaces.*

Proof: We only prove the result for $\ell_\infty(\Delta_g, B)$ and the rest could be demonstrated in the similar way.

Let $v = (v_n) \in \ell_\infty(\Delta_g, B)$ and let $\ell_\infty(\Delta_g, B)$ be considered as sequence space with linear topology.

Define $P_n : \ell_\infty(\Delta_g, B) \to \mathbf{C}$ by $P_n(v) = \Delta_g v_n$.

Now we show that P_n is continuous, for this we will show that P_n is bounded.

$$| P_n(v) = | \Delta_g v_n | = 1 \cdot | \Delta_g v_n | .$$

But $b_{n,k} \ge \beta$ implies that $1 \le \left(\frac{1}{\beta} \right) b_{nk}$. Therefore, we have

$$| P_n(v) | \le \left(\frac{1}{\beta} \right) b_{nk} \, | \Delta_g v_n | \tag{2}$$

$$\begin{aligned} &= \left(\frac{1}{\beta} \right) | b_{nk} | \, | \Delta_g v_n | \ (since\ b_{nk} > 0) \\ &= \left(\frac{1}{\beta} \right) | b_{nk} \Delta_g v_n | \\ &= \left(\frac{1}{\beta} \right) \{ | b_{1k} v_1 | + \sup_n | b_{nk} \Delta_g v_n | \} \\ &= \left(\frac{1}{\beta} \right) \{ | b_{1k} v_1 | + \| b_{nk} \Delta_g v_n \| \} = \left(\frac{1}{\beta} \right) \| v \|_{(\Delta,B)} . \end{aligned}$$

Hence, $P_n(v)$ is bounded and continuous. Therefore, $\ell_\infty(\Delta_g, B)$ is a K-space.

We now state the following result without proof.

Theorem 2.6: *The sets $\ell_\infty(\Delta_g, B)$, $\ell_1(\Delta_g, B)$, $\ell_p(\Delta_g, B)$, $c(\Delta_g, B)$ and $c_0(\Delta_g, B)$ are Banach spaces.*

Theorem 2.7: *The sets $\ell_\infty(\Delta_g, B)$, $\ell_1(\Delta_g, B)$, $\ell_p(\Delta_g, B)$, $c(\Delta_g, B)$ and $c_0(\Delta_g, B)$ are F spaces.*

Proof: We only prove the result for $\ell_\infty(\Delta_g, B)$ and the rest could be demonstrated in the similar way.

As $\ell_\infty(\Delta_g, B)$ is considered to have a linear topology, hence it would be linear. It is earlier shown that $\ell_\infty(\Delta_g, B)$ is assumed as normed linear space and also complete. It is known that all normed linear space is metric space. Hence, $\ell_\infty(\Delta_g, B)$ is a complete linear space. Consequently, $\ell_\infty(\Delta_g, B)$ is a F-space.

Corollary 2.8: $\ell_\infty(\Delta_g, B)$ *will be an FK-space if $\ell_\infty(\Delta_g, B)$ is assumed to be K-space and complete linear metric space.*

Corollary 2.9: *The spaces $\ell_1(\Delta_g, B)$, $\ell_p(\Delta_g, B)$, $c(\Delta_g, B)$ and $c_0(\Delta_g, B)$ are FK-space.*

Theorem 2.10: *The spaces $\ell_\infty(\Delta_g, B)$, $\ell_1(\Delta_g, B)$, $\ell_p(\Delta_g, B)$, $c(\Delta_g, B)$ and $c_0(\Delta_g, B)$ are BK spaces.*

Proof: As the spaces are considered as FK-spaces and their topology is assumed to Normable. Then they are known as BK-spaces.

Definition 2.1: *Let $\ell_{bs}(\Delta_g, B)$ be bounded series in the difference and Köthe matrix be as follows:*

$$\ell_{bs}(\Delta_g, B) = \left\{ v = (v_n) \in \omega : \| v \|_\kappa^{bs} = \sup_{m \in \mathbf{N}} \left| \sum_{n=0}^{m} \Delta_g v_n b_{nk} \right| < \infty \text{ for each } k \in \mathbf{N} \right\}$$

It can be proven that the space $\ell_{bs}(\Delta_g, B)$ is an FK-space and also BK-space considering the norm $||\cdot||_\kappa^{bs}$ given as

$$||v||_\kappa^{bs} = \sup_{m \in \mathbf{N}} \left| \sum_{n=0}^{m} \Delta_g v_n b_{nk} \right| \forall k \in \mathbf{N}.$$

Theorem 2.11: *Let us assume that $1 \le p < \infty$, then*

(i) $\ell_1(\Delta_g, B) \subset \ell_{bs}(\Delta_g, B) \subset \ell_\infty(\Delta_g, B)$.

(ii) Let $\{b_{nk}\}_{n \in \mathbf{N}} \in \ell_p$ for each $k \in \mathbf{N}$, then $\ell_\infty(\Delta_g, B) \subset \ell_p(\Delta_g, B)$.

Proof: (i) To prove $\ell_1(\Delta_g, B) \subset \ell_{bs}(\Delta_g, B)$, let $v \in \ell_1(\Delta_g, B)$, then for each $k \in \mathbf{N}$, we have

$$\sum_n |\Delta_g v_n b_{nk}| < \infty.$$

But

$$\left| \sum_{n=0}^{m} \Delta_g v_n b_{nk} \right| \le \sum_{n=0}^{m} |\Delta_g v_n b_{nk}| < \infty.$$

Now consider supremum over $m \in \mathbf{N}$, we have

$$\sup_{m \in \mathbf{N}} \left| \sum_{n=0}^{m} \Delta_g v_n b_{nk} \right| < \infty.$$

By the definition of $\ell_{bs}(\Delta_g, B)$, we have

$$v \in \ell_{bs}(\Delta_g, B) \Rightarrow \ell_1(\Delta_g, B) \subset \ell_{bs}(\Delta_g, B). \tag{3}$$

Further, to prove $\ell_{bs}(\Delta_g, B) \subset \ell_\infty(\Delta_g, B)$, we let $v \in \ell_{bs}(\Delta_g, B)$. Then, we have

$$\begin{aligned} \| v \|_\kappa^{bs} &= \left\{ \sup_{m\in\mathbf{N}} \left| \sum_{n=0}^{m} \Delta_g v_n b_{nk} \right| < \infty \textit{ for each } k \in \mathbf{N} \right\} \\ &= \left| \sum_{n=0}^{m} \Delta_g v_n b_{nk} \right| < \infty. \end{aligned}$$

Let $L \in \mathbf{R}^+$ such that $\left| \sum_{n=0}^{m} \Delta_g v_n b_{nk} \right| \le L$ for each $k \in \mathbf{N}$, then

$$\begin{aligned} | \Delta_g v_m bmk | &= \left| \sum_{n=0}^{m} \Delta_g v_n b_{nk} - \sum_{n=0}^{m-1} \Delta_g v_n b_{nk} \right| \\ &\le \left| \sum_{n=0}^{m} \Delta_g v_n b_{nk} \right| + \left| \sum_{n=0}^{m-1} \Delta_g v_n b_{nk} \right| \\ &< L + L = 2L. \end{aligned}$$

Considering least upper bound over $m \in \mathbf{N}$, we get $\sup_{n\in\mathbf{N}} | \Delta_g v_m b_{mk} | < \infty$.

Therefore, we conclude that

$$v \in \ell_\infty(\Delta_g, B) \Rightarrow \ell_{bs}(\Delta_g, B) \subset \ell_\infty(\Delta_g, B). \tag{4}$$

From Equation (3) and (4), we get

$$\ell_1(\Delta_g, B) \subset \ell_{bs}(\Delta_g, B) \subset \ell_\infty(\Delta_g, B).$$

(ii) Let $\{b_{nk}\}_{n\in\mathbf{N}} \in \ell_p$ for some $k \in \mathbf{N}$. Also, suppose that $v = (v_n) \in \ell_\infty(\Delta_g)$, then

$$\| v \|_\infty = \left\{ \sum_{n=1}^{\infty} | \Delta_g v_n |^p \right\}^{\frac{1}{p}} < \infty.$$

Since $\{b_{nk}\}_{n\in\mathbf{N}} \in \ell_p$, this yields that

$$\left(\sum_{n=1}^{\infty} | b_{nk} |^p \right)^{\frac{1}{p}} < \infty.$$

Now,

$$\begin{aligned} \sum_{n=1}^{\infty} | \Delta_g v_n b_{nk} |^p &= \sum_{n=1}^{\infty} | \Delta_g v_n |^p | b_{nk} |^p \\ &= \sum_{n=1}^{\infty} | \Delta_g v_n |^p \sum_{n=1}^{\infty} | b_{nk} |^p \\ &= \| v \|_\infty^p \sum_{n=1}^{\infty} | b_{nk} |^p \\ &< \infty. \end{aligned}$$

Therefore, $\left(\sum_{n=1}^{\infty} | b_{nk} |^p \right)^{\frac{1}{p}} < \infty$. Consequently, we have $\ell_\infty(\Delta_g, B) \subset \ell_p(\Delta_g, B)$.

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References

[1] Kothe, G., *Topological vector spaces* (Springer-Verlag, New York), (1969).
[2] Mursaleen, M., *Applied summability methods*, Springer Cham, Heidelberg, (2014).
[3] Ganie, A. H., Sheikh, N. A., *On the sequence space l(p, s) and some matrix transformations*, Nonlinear Functional Analysis and Applications, 18 (2), (2013), 253–258.
[4] Ganie, A. H., Lone, S. A., Akhter, A., *Generalised Cesaro difference sequence space of non-absolute type*, EKSAKTA, 1 (2), (2020), 147–153.
[5] Ray, S, *Difference sequence spaces and matrix methods*, J. Science Engg. & Tech. 10 (1), (2014),73–78.
[6] Maddox, I. J., *Infinite matrices of operators*, Lecture Notes in Math., Infinite matrices of operators, Berlin-Heidelberg-New York: Springer-Verlag, (1980).
[7] Albaidani, M. M., Srivastava, H. M., Ganie, A. H., *Notion of non-absolute family of spaces*, Int. J. Nonlinear Anal. Appl. 14 (1), (2023), 345–354.
[8] Albaidani, M. M., Ganie, A. H., *New Type of Paranormed Behaviour of Spaces*, Journal of Applied Mathematics and Computation, 5 (4), 2021, 273–276.
[9] Kizmaz, H., *On certain sequence spaces*, Canad. Math. Bull 24 (2), (1981), 169–176.
[10] Robinson, A., *On Functional Transformation and Summability* (Proc. Springer Cham Heidelberg).
[11] Ganie, A. H., *Generalized difference sequence spaces*, Adv. Stud. Contemp. Math. (Kyungshang), 30 (2), (2020), 197–206.
[12] Tripathy, B. C., *A New type of difference sequence spaces*, International Journal of Science & Technology, 1(1), (2006), 11–14.
[13] Colac, R., Et, M., *On Koothe duals of generalized difference sequence spaces*, Soochow J. Math, 21 (4), (1995), 377–386.
[14] Šalát, T., Tripathy, B. C., Ziman, M., *On some properties of I-convergence*. Tatra Mt. Math. Publ., 28 (2), (2004), 279–286.
[15] Ganie, A. H., Fathima, D., *On some new scenario of* Δ*-spaces*, J. Nonlinear Sci. Appl., 14, (2021), 163–167.
[16] Fathima, D. Albaidani, M. M., Ganie, A. H., Akhter, A., *New structure of Fibonacci numbers using concept of* Δ*-operator*, Journal of Mathematics and Computer Science, 26(2), (2022), 101–112.
[17] Ganie, A. H., Mobin, A., Sheikh, N. A., Jalal, T., *New type of Riesz sequence space of non-absolute type*, J. Appl. Comput. Math., 5 (1), (2016), 1–4.
[18] Sheikh, N. A., Ganie, A. H., *A new type of sequence space of non-absolute type and matrix transformation*, WSEAS Transaction of Math., 8 (12), (2013), 852–859.
[19] Sheikh, N. A., Jalal T., Ganie, A. H., *New type of sequence spaces of non-absolute type and some matrix transformations*, Acta Mathematica Academiae Paedagogicae Nyregyháziensis, 29, (2013), 51–66.