Results in Nonlinear Analysis 6 (2023) No. 1, 98–106
https://doi.org/10.31838/rna/2023.06.01.009
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

The fractional integrodifferential operator and its univalence and boundedness features according to Pre-Schwarzian derivative structure

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Abstract

Complex-valued regular functions that are normalized in the open unit disk are vastly studied. The current study introduces a new fractional integrodifferential (non-linear) operator. Based on the pre-Schwarzian derivative, certain appropriate stipulations on the parameters included in this constructed operator to be univalent and bounded are investigated and determined.

Keywords: Regular function, Locally univalent, Pre-Schwarzian derivative, Fractional calculus.

2010 Mathematics Subject Classifications: 30C45, 30C50, 30C10.

1. Introduction

Over the past few decades, the theme of fractional calculus (FRC) has been widely, studied principally due to its significant implementations in mathematics and other fields related to it. Specifically, analysis (complex and mathematical) considerably evolved from FRC, which includes the incipient notions and analysis' techniques [12], [13]. The foundations of the FRC theme were posed by Leibnitz in 1695. Its main concern is to study the extension of order derivatives and integrals to fractional derivatives and fractional integrals.

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Received January 11, 2023; Accepted February 17, 2023; Online March 25, 2023

FRC has recently attracted the concern of the analytical community to the geometric complex function theory (GCFT). This great interest is due to its use as a valuable tool in researching a variety of operators with successful implementations in GCFT. In reality, Srivastava and Owa have introduced lots of contributions to developing on the theory FRC in the complex unit disk, such as [18], [21], and [22]. Since then, many authors have provided to this area. For instance, Amsheri and Zharkova [3], Farzana *et al* [8], Ghanim and Al-Janaby [9], [10], [11].

Let $\Delta = \{z \in \mathcal{C} : |z| < 1\}$ indicate the unit disc in $\mathcal{C}$ "complex plane". The class of all regular functions in Δ is denoted by $\mathcal{E}(\Delta)$. Consider Ω_Δ the subclass of $\mathcal{E}(\Delta)$ that includes normalized regular functions f of the formulas:

$$f(z) = z + \sum_{\kappa=2}^{\infty} a_\kappa z^\kappa, \qquad (z \in \Delta). \tag{1}$$

Besides, let Λ_Δ identify the subclass of Ω_Δ involving univalent functions. Denote by $\mathcal{C}^{\mathcal{V}}$ and $\mathcal{S}^*$ consecutively the subclasses of Ω_Δ involving starlike and convex functions. Functions $f \in \mathcal{C}^{\mathcal{V}}$ map Δ onto convex domain, while $f \in \mathcal{S}^*$ whenever $f(\Delta)$ starlike domains with respect to the origin. Analytically, $f \in \mathcal{C}^{\mathcal{V}}$ if $0 < \Re\left(1 + \dfrac{zf''(z)}{f'(z)}\right)$, while $f \in \mathcal{S}^*$ if $0 < \Re\left(\dfrac{zf'(z)}{f(z)}\right)$. Moreover, for $0 \le \varepsilon < 1$, we denote the subclasses of Ω_Δ consisting of convex functions and starlike functions of order ε by $\mathcal{C}^{\mathcal{V}}_\varepsilon$ and $\mathcal{S}^*_\varepsilon$, consecutively. These geometric subclasses achieve the following series of proper inclusions: $\mathcal{C}^{\mathcal{V}}_\varepsilon \subset \mathcal{S}^*_\varepsilon \subset \Lambda_\Delta$. Moreover, they are regularly acquainted by

$$\mathcal{C}^{\mathcal{V}}_\varepsilon = \left\{ f \in \Omega_\Delta : \varepsilon < \Re\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}, \tag{2}$$

and

$$\mathcal{S}^*_\varepsilon = \left\{ f \in \Omega_\Delta : \varepsilon < \Re\left(\frac{zf'(z)}{f(z)}\right)\right\}. \tag{3}$$

Obviously, for $\varepsilon = 0$, then $\mathcal{C}^{\mathcal{V}}_\varepsilon$ and $\mathcal{S}^*_\varepsilon$ coincide with $\mathcal{C}^{\mathcal{V}}$ and $\mathcal{S}^*$, sequentially, [14]. The significant connection between the subclasses $\mathcal{C}^{\mathcal{V}}_\varepsilon$ and $\mathcal{S}^*_\varepsilon$, called "Alexander-sort", which achieves $f \in \mathcal{C}^{\mathcal{V}}_\varepsilon$ if and only if $zf' \in \mathcal{S}^*_\varepsilon$, [1]. The function f is obtained from g as follows:

$$f(z) = \Im[g](z) = \int_0^z \frac{g(\tau)}{\tau} d\tau = \int_0^1 \frac{g(\mathrm{t}z)}{\mathrm{t}} d\mathrm{t}, \tag{4}$$

where $\Im[g]$ is called the Alexander transform of $g \in \mathfrak{L}_\Delta$. Therefore, $g \in \mathcal{S}^*_\varepsilon$ if and only if $\Im[g] \in \mathcal{C}^{\mathcal{V}}_\varepsilon$. Since the function $g(z) = z + \sum_{\kappa=2}^{\infty} a_\kappa z^\kappa$ is transformed by = to $\Im[g](z) = z + \sum_{\kappa=2}^{\infty} a_\kappa \dfrac{z^\kappa}{\kappa}$, it is expected that the function $\Im[g]$ is closer to similarities than g, see [2] and [6].

Interesting investigations continued into certain subclasses of Ω_Δ. This includes the class $\mathfrak{L}_\Delta$ consisting of locally univalent functions in Δ, namely,

$$\mathfrak{L}_\Delta = \{ f \in \Omega_\Delta : f'(z) \ne 0, z \in \Delta \} \tag{5}$$

is a vector space over $\mathcal{C}$ in the sense of the Hornich operations [15]. For $f \in \mathfrak{L}_\Delta$, the term "pre-Schwarzian derivative" $\mathcal{P}_f(z)$ is the logarithmic derivative of f and given by the following formula:

$$\mathcal{P}_f(z) = \frac{f''(z)}{f'(z)}. \tag{6}$$

Furthermore, the norm of $\mathcal{P}_f$ is stated as:

$$\left\|\mathcal{P}_f\right\| = \sup_{|z|<1}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right|, \tag{7}$$

see [4]. The pre-Schwarzian derivative is important in the theory of Teichmüller spaces and has a number of implementations in the theory of locally univalent functions. This norm is widely utilized in the study of geometric features of such functions. Specifically, it can be utilized to acquire either necessary or suitable stipulations for the global univalence or to gain certain geometric stipulations on the range of the function. Also, it is well known that any univalent regular transformation in Δ achieves the sharp inequality $|\mathcal{P}_f(z)| \le \dfrac{6}{1-|z|^2}$, $z \in \Delta$. Further, in view of Becker's univalence criterion, every regular function f in Δ with $\left\|\mathcal{P}_f\right\| \le 1$ is univalent in Δ. Conversely $\left\|\mathcal{P}_f\right\| \le 6$ is achieved if f is univalent, [4]. Recently, several authors have analyzed and introduced the norm estimates for typical subclasses of univalent functions, [19] and [20].

The following theorem has salutary tools discussing the major outcomes:

Theorem 1.1. *For* $f \in \mathfrak{L}_\Delta$,

1. If $\left\|\mathcal{P}_f\right\| \le 1$, then f is univalent,
2. If $\left\|\mathcal{P}_f\right\| \le 2$, then f is bounded.

The constants are sharp

The first part is posed by Becker ([4], p. 36, corollary 4.1), while the sharpness of the constant 1 is due to Becker and Pommerenke [5]. The second part is offered by Kim and Sugawa. [16].

Theorem 1.2. *For* $0 \le \varepsilon < 1$ *and* $f \in \Omega_\Delta$, *then*

1. If $f \in \mathcal{C}_\varepsilon^\nu$, acquire $\left\|\mathcal{P}_f\right\| \le 4(1-\varepsilon)$,
2. If $f \in \mathcal{S}_\varepsilon^*$, obtain $\left\|\mathcal{P}_f\right\| \le 6-4\varepsilon$.

This theorem is attributed to Yamashita ([24], p. 219, Theorems 1 and 2).

On the other hand, the following fractional calculus (differential and integral) operators in the sense of Srivastava and Owa operators [22]: for function f, the fractional derivative of order α is given by

$$\mathfrak{D}_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_0^z \frac{f(\tau)}{(z-\tau)^\alpha}\,d\tau, \qquad 0 \le \alpha < 1, \tag{8}$$

where function f is regular in a simply connected region of $\mathcal{C}$, including the origin, and the multiplicity of $(z-\tau)^{-\alpha}$ is extracted by demanding $\log(z-\tau)$ to be real when $0 < (z-\tau)$. Whilst the fractional integral of order α is given by

$$\mathfrak{I}_z^\alpha\, f(z) = \frac{1}{\Gamma(\alpha)}\int_0^z f(\tau)(z-\tau)^{\alpha-1}\,d\tau, \qquad 0 < \alpha, \tag{9}$$

where function f is regular in a simply connected region of $\mathcal{C}$, including the origin, and the multiplicity of $(z-\tau)^{\alpha-1}$ is removed by demanding $\log(z-\tau)$ to be real when $0 < (z-\tau)$.

The following lemma gives analogous formulations to the above concepts of fractional operators:

Lemma 1.3. [22]

1. $\mathfrak{D}_z^\alpha\, z^\kappa = \dfrac{\Gamma(\kappa+1)}{\Gamma(\kappa-\alpha+1)}\, z^{\kappa-\alpha}$, $-1 < \kappa$, $0 \le \alpha < 1$,

2. $\mathfrak{I}_z^\alpha\, z^\kappa = \dfrac{\Gamma(\kappa+1)}{\Gamma(\kappa+\alpha+1)}\, z^{\kappa+\alpha}$, $-1<\kappa,\ 0<\alpha$.

In this paper, a new fractional integrodifferential operator is provided. Related to the pre-Schwarzian derivative, certain appropriate stipulations on the parameters included in this constructed operator to be univalent and bounded are considered and discussed.

2. Fractional Integrodifferential Operator $\mathfrak{S}_\mu(z)$

This section presents, for $f \in \Omega_\Delta$, a new fractional integrodifferential (non-linear) operator based on fractional binomial expansion and fractional differential formal operator in the sense of Srivastava and Owa operators.

In terms of fractional binomial expansion, $0 \le \alpha < 1$, we consider a new fractional regular function for $f \in \Omega_\Delta$ as:

$$\begin{aligned} \vartheta_f(z) = \frac{z\, f^\alpha(z)}{\Gamma(\alpha+2)} &= \frac{z}{\Gamma(\alpha+2)}\left[z^\alpha + \alpha\, a_2\, z^{\alpha+1} + \left(\alpha\, a_3 + \frac{\alpha(\alpha-1)}{2!}a_2^3\right) z^{\alpha+2} + \ldots\right] \\ &= \frac{z^{\alpha+1}}{\Gamma(\alpha+2)} + \sum_{\kappa=2}^{\infty} \alpha\, \frac{\delta_\kappa(\alpha)}{\Gamma(\alpha+2)}\, z^{\alpha+\kappa}, \end{aligned} \tag{10}$$

where $\delta_\kappa(\alpha)$ is the coefficients depending on a_κ of *f*.

In view of the fractional differential formula in the sense of Srivastava and Owa operators given by Theorem 3, the new fractional function ϑ_f (10) yields the following fractional differential operator:

$$\begin{aligned} \mathfrak{D}_z^\alpha\, \vartheta_f(z) &= \frac{\mathfrak{D}_z^\alpha\, z^{\alpha+1}}{\Gamma(\alpha+2)} + \sum_{\kappa=2}^{\infty} \alpha\, \frac{\delta_\kappa(\alpha)}{\Gamma(\alpha+2)}\, \mathfrak{D}_z^\alpha\, z^{\alpha+\kappa} \\ &= z + \sum_{\kappa=2}^{\infty} \frac{\alpha\,(\alpha+\kappa)!}{(\alpha+1)!\,\kappa!}\, \delta_\kappa(\alpha)\, z^\kappa, \qquad 0 \le \alpha < 1. \end{aligned} \tag{11}$$

Remark 2.1. *Notice that*

1. $\mathfrak{D}_z^0\, \vartheta_f(z) = z$,
2. $\mathfrak{D}_0^\alpha\, \vartheta_f(0) = 0$,
3. $(\mathfrak{D}_0^\alpha \vartheta_f)'(0) = 1$,
4. $\mathfrak{I}_z^\alpha \mathfrak{D}_z^\alpha \vartheta_f(z) = \vartheta_f(z)$.

Then, we impose the following new modified classes of convex and starlike functions correlated with the fractional operator (11):

Let $\mathcal{C}_\varepsilon^\mathcal{V}(\alpha)$ denote the class of functions $f \in \Omega_\Delta$ which achieve the condition

$$\varepsilon < \Re\left(1 + \frac{z(\mathfrak{D}_z^\alpha \vartheta_f)''(z)}{(\mathfrak{D}_z^\alpha \vartheta_f)'(z)}\right), \tag{12}$$

where $0 \le \varepsilon < 1$, $0 \le \alpha < 1$ and $z \in \Delta$. Clearly, $\mathfrak{D}_z^\alpha \vartheta_f(z) \in \mathcal{C}_\varepsilon^\mathcal{V}$.

Also, let $\mathcal{S}_{\varepsilon}^{*}(\alpha)$ be the class of functions $f \in \Omega_{\Delta}$ achieving

$$\varepsilon < \Re\left(\frac{z(\mathfrak{D}_z^{\alpha}\vartheta_f)'(z)}{\mathfrak{D}_z^{\alpha}\vartheta_f(z)}\right). \tag{13}$$

where $0 \le \varepsilon < 1$, $0 \le \alpha < 1$ and $z \in \Delta$. Clearly, $\mathfrak{D}_z^{\alpha}\vartheta_f(z) \in \mathcal{S}_{\varepsilon}^{*}$. Further, it is obvious that $f \in \mathcal{C}_{\varepsilon}^{\mathcal{V}}(\alpha)$ if and only if $zf' \in \mathcal{S}_{\varepsilon}^{*}(\alpha)$.

Therefore, based on the study in [23] and the fractional operator given by (11), let's introduce a fractional integrodifferential (non-linear) operator $\mathfrak{S}_{\mu}(z) : \Omega_{\Delta}^{\mu} \to \Omega_{\Delta}$ as:

$$\mathfrak{S}_{\mu}(z) = \int_0^z \prod_{\iota=1}^{\mu}\left((\mathfrak{D}_z^{\alpha}\vartheta_{f_\iota})'(\tau)\right)^{\eta_\iota}\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_{\sigma_\iota}(\tau)}{\tau}\right)^{\ell_\iota} d\tau, \tag{14}$$

where f_ι, $\sigma_\iota \in \Omega_{\Delta}$, $0 \le \eta_\iota$, ℓ_ι for $\iota = 1,2,\ldots,\mu$ and $\mu \in \mathbb{N}$.

Remark 2.2. *The non-linear operator $\mathfrak{S}_{\mu}(z)$ (14) serves as a new generalization for the following operators by means of various choices of the parameters involved.*

1. For $\ell_\iota = 0$, where $\iota = 1,\ldots,\mu$, yields the operator

$$\Phi_{\mu}(z) = \int_0^z \prod_{\iota=1}^{\mu}((\mathfrak{D}_z^{\alpha}\vartheta_{f_\iota})'(\tau))^{\eta_\iota}\, d\tau, \tag{15}$$

2. For $\eta_\iota = 0$, where $\iota = 1,\ldots,\mu$, gains the operator

$$\Psi_{\mu}(z) = \int_0^z \prod_{\iota=1}^{\mu}\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_{\sigma_\iota}(\tau)}{\tau}\right)^{\ell_\iota} d\tau, \tag{16}$$

3. For $\mu = 1$, $\eta_1 = \eta$, $\ell_1 = \ell$ and $f_1 = \sigma_1 = f$, acquires the operator

$$\Omega(z) = \int_0^z \left((\mathfrak{D}_z^{\alpha}\vartheta_f)'(\tau)\right)^{\eta}\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_f(\tau)}{\tau}\right)^{\ell} d\tau, \tag{17}$$

4. For $\mu = 1$, $\eta_1 = 1$, $\ell_1 = 1$ and $f_1 = \sigma_1 = f$, obtains the operator

$$\Sigma(z) = \int_0^z \left((\mathfrak{D}_z^{\alpha}\vartheta_f)'(\tau)\right)\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_f(\tau)}{\tau}\right) d\tau. \tag{18}$$

3. Univalence and boundedness of $\mathfrak{S}_{\mu}(z)$

This section investigates certain appropriate stipulations for univalence and boundedness of new fractional integrodifferential operator $\mathfrak{S}_{\mu}(z)$ given by (14).

An implementation of Theorem 1.1 and 1.2 to the new fractional operator (14) acquires the following major outcome.

Theorem 3.4. *For $0 \le \eta_\iota$, ℓ_ι, $\iota = 1,2,\ldots,\mu$, let f_ι, $\sigma_\iota \in \mathcal{S}_{\varepsilon_\iota}^{*}(\alpha)$. If $\mathfrak{S}_{\mu}(z)$ is locally univalent in Δ and achieves*

$$\sum_{\iota=1}^{\mu}\left[\eta_\iota\left(3 - 2\varepsilon_\iota\right) + 2\ell_\iota\left(1 - \varepsilon_\iota\right)\right] \le \frac{1}{2}, \tag{19}$$

then $\mathfrak{S}_{\mu}(z)$ is univalent. Further, if

$$\sum_{\iota=1}^{\mu}\left[\eta_{\iota}\left(3-2\varepsilon_{\iota}\right)+2\ell_{\iota}\left(1-\varepsilon_{\iota}\right)\right]\leq 1, \tag{20}$$

then $\mathfrak{S}_{\mu}(z)$ is bounded.

Proof. In light of (7) and (14), we acquire

$$\left\|\mathcal{P}_{\mathfrak{S}_{\mu}}\right\| = \sup_{|z|<1}\left(1-|z|^2\right)\left|\frac{\mathfrak{S}''_{\mu}(z)}{\mathfrak{S}'_{\mu}(z)}\right|$$

$$= \sup_{|z|<1}\left(1-|z|^2\right)\left|\frac{\left(\int_0^z \prod_{\iota=1}^{\mu}\left((\mathfrak{D}_z^{\alpha}\vartheta_{f_\iota})'(\tau)\right)^{\eta_\iota}\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_{\sigma_\iota}(\tau)}{\tau}\right)^{\ell_\iota}d\tau\right)''}{\left(\int_0^z \prod_{\iota=1}^{\mu}\left((\mathfrak{D}_z^{\alpha}\vartheta_{f_\iota})'(\tau)\right)^{\eta_\iota}\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_{\sigma_\iota}(\tau)}{\tau}\right)^{\ell_\iota}d\tau\right)'}\right|$$

$$= \sup_{|z|<1}\left(1-|z|^2\right)\left|\frac{\eta_1\left((\mathfrak{D}_z^{\alpha}\vartheta_{f_1})''(z)\right)\left((\mathfrak{D}_z^{\alpha}\vartheta_{f_1})'(z)\right)^{\eta_1-1}\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_{\sigma_1}(z)}{z}\right)^{\ell_1}}{\left((\mathfrak{D}_z^{\alpha}\vartheta_{f_1})'(z)\right)^{\eta_1}\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_{\sigma_1}(z)}{z}\right)^{\ell_1}}\right.$$

$$\times\frac{\left((\mathfrak{D}_z^{\alpha}\vartheta_{f_2})'(z)\right)^{\eta_2}\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_{\sigma_2}(z)}{z}\right)^{\ell_2}\ldots\left((\mathfrak{D}_z^{\alpha}\vartheta_{f_\mu})'(z)\right)^{\eta_\mu}\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_{\sigma_\mu}(z)}{z}\right)^{\ell_\mu}}{\left((\mathfrak{D}_z^{\alpha}\vartheta_{f_2})'(z)\right)^{\eta_2}\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_{\sigma_2}(z)}{z}\right)^{\ell_2}\ldots\left((\mathfrak{D}_z^{\alpha}\vartheta_{f_\mu})'(z)\right)^{\eta_\mu}\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_{\sigma_\mu}(z)}{z}\right)^{\ell_\mu}}$$

$$+\frac{\ell_1\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_{\sigma_1}(z)}{z}\right)'\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_{\sigma_1}(z)}{z}\right)^{\ell_1-1}\left((\mathfrak{D}_z^{\alpha}\vartheta_{f_1})'(z)\right)^{\eta_1}\left((\mathfrak{D}_z^{\alpha}\vartheta_{f_2})'(z)\right)^{\eta_2}\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_{\sigma_2}(z)}{z}\right)^{\ell_2}}{\left((\mathfrak{D}_z^{\alpha}\vartheta_{f_1})'(z)\right)^{\eta_1}\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_{\sigma_1}(z)}{z}\right)^{\ell_1}\left((\mathfrak{D}_z^{\alpha}\vartheta_{f_2})'(z)\right)^{\eta_2}\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_{\sigma_2}(z)}{z}\right)^{\ell_2}}$$

$$\left.\times\frac{\ldots\left((\mathfrak{D}_z^{\alpha}\vartheta_{f_\mu})'(z)\right)^{\eta_\mu}\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_{\sigma_\mu}(z)}{z}\right)^{\ell_\mu}}{\ldots\left((\mathfrak{D}_z^{\alpha}\vartheta_{f_\mu})'(z)\right)^{\eta_\mu}\left(\frac{\mathfrak{D}_z^{\alpha}\vartheta_{\sigma_\mu}(z)}{z}\right)^{\ell_\mu}}+\ldots\right|$$

$$= \sup_{|z|<1}\left(1-|z|^2\right)\left|\eta_1 \frac{(\mathfrak{D}_z^\alpha \vartheta_{f_1})''(z)}{(\mathfrak{D}_z^\alpha \vartheta_{f_1})'(z)} + \ell_1 \frac{\left(\frac{\mathfrak{D}_z^\alpha \vartheta_{\sigma_1}(z)}{z}\right)'}{\left(\frac{\mathfrak{D}_z^\alpha \vartheta_{\sigma_1}(z)}{z}\right)} + \ldots + \eta_\mu \frac{(\mathfrak{D}_z^\alpha \vartheta_{f_m u})''(z)}{(\mathfrak{D}_z^\alpha \vartheta_{f_\mu})'(z)} + \ell_\mu \frac{\left(\frac{\mathfrak{D}_z^\alpha \vartheta_{\sigma_\mu}(z)}{z}\right)'}{\left(\frac{\mathfrak{D}_z^\alpha \vartheta_{\sigma_\mu}(z)}{z}\right)}\right|$$

$$= \sup_{|z|<1}\left(1-|z|^2\right)\left|\sum_{\iota=1}^{\mu}\eta_\iota \frac{(\mathfrak{D}_z^\alpha \vartheta_{f_\iota})''(z)}{(\mathfrak{D}_z^\alpha \vartheta_{f_\iota})'(z)} + \sum_{\iota=1}^{\mu}\ell_\iota \frac{\left(\frac{\mathfrak{D}_z^\alpha \vartheta_{\sigma_\iota}(z)}{z}\right)'}{\left(\frac{\mathfrak{D}_z^\alpha \vartheta_{\sigma_\iota}(z)}{z}\right)}\right|. \tag{21}$$

Since $\sigma_\iota \in \mathcal{S}_{\varepsilon_\iota}^*(\alpha)$, this yields $\mathfrak{D}_z^\alpha \vartheta_{\sigma_\iota}(z) \in \mathcal{S}_{\varepsilon_\iota}^*$. In view of the Alexander-sort relation, it follows that there is $\mathfrak{D}_z^\alpha \vartheta_{\varphi_\iota} \in \mathcal{C}_\varepsilon^\nu$, where $\varphi_\iota \in \mathcal{C}_\varepsilon^\nu(\alpha)$ such that

$$\mathfrak{D}_z^\alpha \vartheta_{\sigma_\iota}(z) = z(\mathfrak{D}_z^\alpha \vartheta_{\varphi_\iota})'(z). \tag{22}$$

In addition, since $f_\iota \in \mathcal{S}_{\varepsilon_\iota}^*(\alpha)$, it follows that $\mathfrak{D}_z^\alpha \vartheta_{f_\iota}(z) \in \mathcal{S}_{\varepsilon_\iota}^*$. Therefore,

$$\begin{aligned}\left\|\mathcal{P}_{\mathfrak{S}_\mu}\right\| &= \sup_{|z|<1}\left(1-|z|^2\right)\left|\sum_{\iota=1}^{\mu}\eta_\iota \frac{(\mathfrak{D}_z^\alpha \vartheta_{f_\iota})''(z)}{(\mathfrak{D}_z^\alpha \vartheta_{f_\iota})'(z)} + \sum_{\iota=1}^{\mu}\ell_\iota \frac{(\mathfrak{D}_z^\alpha \vartheta_{\varphi_\iota})''(z)}{(\mathfrak{D}_z^\alpha \vartheta_{\varphi_\iota})'(z)}\right| \\ &\le \sum_{\iota=1}^{\mu}\eta_\iota \sup_{|z|<1}\left(1-|z|^2\right)\left|\frac{(\mathfrak{D}_z^\alpha \vartheta_{f_\iota})''(z)}{(\mathfrak{D}_z^\alpha \vartheta_{f_\iota})'(z)}\right| + \sum_{\iota=1}^{\mu}\ell_\iota \sup_{|z|<1}\left(1-|z|^2\right)\left|\frac{(\mathfrak{D}_z^\alpha \vartheta_{\varphi_\iota})''(z)}{(\mathfrak{D}_z^\alpha \vartheta_{\varphi_\iota})'(z)}\right| \\ &= \sum_{\iota=1}^{\mu}\eta_\iota \left\|\mathcal{P}_{f_\iota}\right\| + \sum_{\iota=1}^{\mu}\ell_\iota \left\|\mathcal{P}_{\varphi_\iota}\right\|.\end{aligned} \tag{23}$$

By an application of Theorem 1.2, this gains

$$\left\|\mathcal{P}_{\mathfrak{S}_\mu}\right\| \le 2\sum_{\iota=1}^{\mu}\left[\eta_\iota\left(3-2\varepsilon_\iota\right) + 2\ell_\iota\left(1-\varepsilon_\iota\right)\right]. \tag{24}$$

From inequality (19), it leads to

$$\left\|\mathcal{P}_{\mathfrak{S}_\mu}\right\| \le 1. \tag{25}$$

Hence, by the first part of Theorem 1.1, $\mathfrak{S}_\mu(z)$ is univalent.

Furthermore, from condition (20), we deduce

$$\left\|\mathcal{P}_{\mathfrak{S}_\mu}\right\| \le 2. \tag{26}$$

Thus, in view of the second part of Theorem 1 $\mathfrak{S}_\mu(z)$ is bounded.

Theorem 3.5. *For $0 \le \eta_\iota,\ \ell_\iota,\ \iota = 1,2,\dots,\mu$, let $f_\iota \in \mathcal{C}^{\nu}_{\varepsilon_\iota}(\alpha)$ and $\sigma_\iota \in \mathcal{S}^{*}_{\varepsilon_\iota}(\alpha)$. If $\mathfrak{S}_\mu(z)$ is locally univalent in Δ and achieves*

$$\sum_{\iota=1}^{\mu}\left[\left(\eta_\iota + \ell_\iota\right)\left(1-\varepsilon_\iota\right)\right] \le \frac{1}{4}, \tag{27}$$

then $\mathfrak{S}_\mu(z)$ is univalent. Moreover, if

$$\sum_{\iota=1}^{\mu}\left[\left(\eta_\iota + \ell_\iota\right)\left(1-\varepsilon_\iota\right)\right] \le \frac{1}{2}, \tag{28}$$

then $\mathfrak{S}_\mu(z)$ is bounded.

Proof. Since $f_\iota \in \mathcal{C}^{\nu}_{\varepsilon_\iota}(\alpha)$, it follows that $\mathfrak{D}^{\alpha}_{z}\,\vartheta_{f_\iota}(z) \in \mathcal{C}_{\varepsilon_\iota}$. From (22), by utilizing inequality (23) and employing the first part of Theorem 2, we conclude

$$\begin{aligned}\left\|\mathcal{P}_{\mathfrak{S}_\mu}\right\| &\le 4\sum_{\iota=1}^{\mu}\left[\left(\eta_\iota\left(1-\varepsilon_\iota\right) + \ell_\iota\left(1-\varepsilon_\iota\right)\right)\right] \\ &= 4\sum_{\iota=1}^{\mu}[(\eta_\iota + \ell_\iota)(1-\varepsilon_\iota)].\end{aligned} \tag{29}$$

Therefore, from condition (27) and in view of the first part of Theorem 1.1, $\mathfrak{S}_\mu(z)$ is univalent.

On the other hand, then, from inequality (28) and by the second part of Theorem 1.1, $\mathfrak{S}_\mu(z)$ is bounded.

Corollary 3.6. *For $0 \le \eta_\iota,\ \ell_\iota,\ \iota = 1,2,\dots,\mu$, let $f_\iota,\ \sigma_\iota \in \mathcal{S}^{*}_{\varepsilon}(\alpha)$. If $\mathfrak{S}_\mu(z)$ is locally univalent in Δ and achieves*

$$\left(6-4\varepsilon\right)\sum_{\iota=1}^{\mu}\eta_\iota + 4\left(1-\varepsilon\right)\sum_{\iota=1}^{\mu}\ell_\iota \le \frac{1}{2}, \tag{30}$$

then $\mathfrak{S}_\mu(z)$ is univalent. Furthermore, if

$$\left(6-4\varepsilon\right)\sum_{\iota=1}^{\mu}\eta_\iota + 4\left(1-\varepsilon\right)\sum_{\iota=1}^{\mu}\ell_\iota \le 1, \tag{31}$$

then $\mathfrak{S}_\mu(z)$ is bounded.

Corollary 3.7. *For $0 \le \eta_\iota,\ \ell_\iota,\ \iota = 1,2,\dots,\mu$, let $f_\iota \in \mathcal{S}^{\nu}_{\varepsilon_\iota}(\alpha)$ and $\sigma_\iota \in \mathcal{S}^{*}_{\varepsilon_\iota}(\alpha)$. If $\mathfrak{S}_\mu(z)$ is locally univalent in Δ and achieves*

$$\left(1-\varepsilon\right)\sum_{\iota=1}^{\mu}\left(\eta_\iota + \ell_\iota\right) \le \frac{1}{4}, \tag{32}$$

then $\mathfrak{S}_\mu(z)$ is univalent. Further, if

$$\left(1-\varepsilon\right)\sum_{\iota=1}^{\mu}\left(\eta_\iota + \ell_\iota\right) \le \frac{1}{2}, \tag{33}$$

then $\mathfrak{S}_\mu(z)$ is bounded.

Competing interests

The authors declare that they have no competing interests.

Acknowledgments

This research was funded by the University of Sharjah, project ID: 2002144093.

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