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Fractional curl with standard fractional vector cross product for a vector pair

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Abstract

In this research fractional curl of an electromagnetic vector field is presented using the new definition of Standard Fractional Vector Cross Product (SFVCP). The properties are further supported with particular case at $\gamma = 1$ which satisfies the standard vector cross product. The new definition is also defined for $\gamma = 0, 0 \le \gamma \le 1, \gamma = 1$. These mentioned concepts makes it geometrically real and visually more presentable. This new definition can be applied in various fields of electromagnetic theory, electrodynamics, elastodynamics, fluid flow etc.

Mathematics Subject Classification (2010): 15A30, 15A63, 26A99, 26B12

Key words and phrases: Vector cross product; Fractional vector cross product; Fractional Curl

1. Introduction

Crowe [1] laid the foundation of vector analysis in 1967. Das [2] extended the idea of vector cross product and defined fractional vector cross product and fractional curl with application in vector field of electromagnetic theory. Later in 2022, Tripathi and Kim [6] presented α - fractional cross product of two vectors in Euclidean 3-space which resembled the definition given by Das [2]. In 2022, Kankarej and Singh [4] defined standard fractional vector cross product of two vectors in Euclidean 3-space which satisfies all the conditions of geometrical reality. In this paper we used definition of [4] i. e. *Standard Fractional Vector Cross Product* in Euclidean 3-space to calculate fractional vector cross product for a vector pair.

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To be precise, this research derives a general expression for obtaining cross product of 2 vectors in R^3 space. We also obtain fractional curl via fractionalized vector cross product in $\bar{k}$ space in R^3 where $\bar{k}$ space is $(\hat{x}_k, \hat{y}_k, \hat{z}_k)$ which is Fourier Transformed for spatial coordinate $(\bar{x}, \bar{y}, \bar{z})$ space in R^3.

2. Main Results

The main focus of this paper is to use the new definition of Standard Fractional Vector Cross Product in Euclidean 3-space [4] to calculate fractional vector cross product for a vector pair. The flow of the paper includes:

1. The explanation of new definition.
2. Graphical representation of the new definition.
3. Derivation of SFVCP for first vector pair.
4. Derivation of SFVCP for second vector pair.
5. Derivation of SFVCP.

3. Standard Fractional Vector Cross Product

Further to the study of fractional cross product in [6], we defined an alternative definition which satisfies the assumption given below:

To be in 3-dimensional space a vector product must be represented in 3-dimension, which is satisfied in new definition.

For $\gamma = 0$, the standard vector cross product is represented in a 2-dimensional plane.

For $0 \le \gamma \le 1$, the standard vector cross product is an arbitrary vector in 3-dimension.

For $\gamma = 1$, the standard vector cross product behaves like a normal vector cross product.

And hence we define:

Definition 3.1: *Let R^3 be the Euclidean 3-space equipped with standard inner product $\langle \cdot,\cdot \rangle$. Let (e_1, e_2, e_3) be standard orthonormal basis of R^3 and $\gamma \in [0,1]$ a real number. Then, for vectors $a = a_1e_1 + a_2e_2 + a_3e_3$, $b = b_1e_1 + b_2e_2 + b_3e_3$ in R^3, the Standard Fractional Vector Cross Product is defined by*

$$\begin{aligned} a \times^{\gamma} b = &\left\{(a_2b_3 - a_3b_2)\sin\left(\frac{\gamma\pi}{2}\right) + (a_2 + a_3)b_1\cos\left(\frac{\gamma\pi}{2}\right) - (b_2 + b_3)a_1\cos\left(\frac{\gamma\pi}{2}\right)\right\}e_1 \\ &+ \left\{(a_3b_1 - a_1b_3)\sin\left(\frac{\gamma\pi}{2}\right) + (a_3 + a_1)b_2\cos\left(\frac{\gamma\pi}{2}\right) - (b_3 + b_1)a_2\cos\left(\frac{\gamma\pi}{2}\right)\right\}e_2 \\ &+ \left\{(a_1b_2 - a_2b_1)\sin\left(\frac{\gamma\pi}{2}\right) + (a_1 + a_2)b_3\cos\left(\frac{\gamma\pi}{2}\right) - (b_1 + b_2)a_3\cos\left(\frac{\gamma\pi}{2}\right)\right\}e_3 \end{aligned} \tag{1}$$

From eqn (1) we have,

$$e_i \times^{\gamma} e_j = \cos\left(\frac{\gamma\pi}{2}\right)e_j + \sin\left(\frac{\gamma\pi}{2}\right)e_k - \cos\left(\frac{\gamma\pi}{2}\right)e_i \tag{2}$$

$$e_j \times^{\gamma} e_i = \cos\left(\frac{\gamma\pi}{2}\right)e_i - \sin\left(\frac{\gamma\pi}{2}\right)e_k - \cos\left(\frac{\gamma\pi}{2}\right)e_j \tag{3}$$

$$e_l \times^{\gamma} e_l = 0 \ for\ l = \{1,2,3\} \tag{4}$$

where (i, j, k) is a cyclic permutation of (1, 2, 3). The equations (2), (3) and (4) are similar to that in [5] and [6].

4. Graphical Representation Of Standard Fractional Vector Cross Product

Figures 1 and 2 shows that the SFVCP represented by (2) and (3) is a curve having values from $x-y$ plane at $\gamma=0$ to z-plane at $\gamma=1$. The movement of the vectors for different values of γ for $e_i \times^{\gamma} e_j$ and for $e_j \times^{\gamma} e_i$ are represented in Figure 1 and 2 respectively.

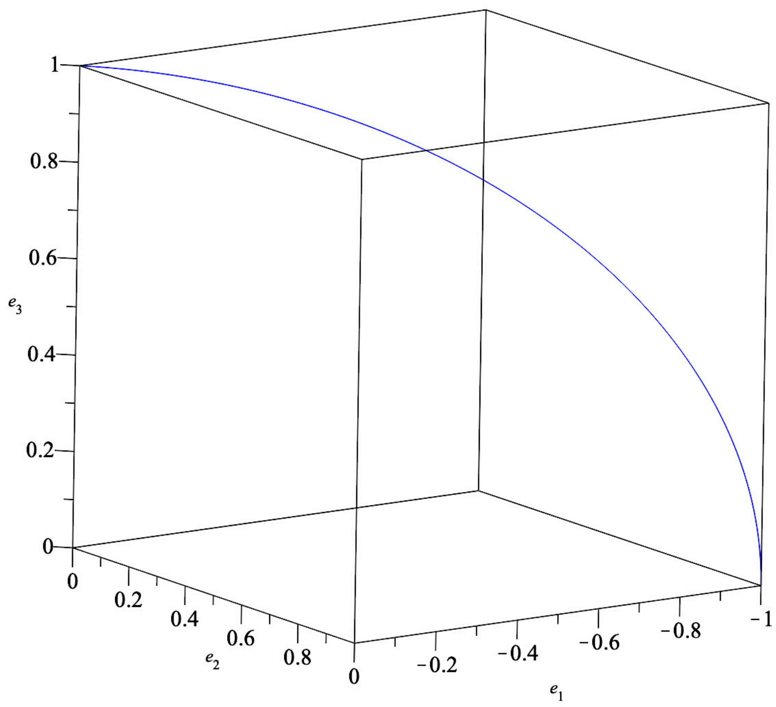

Figure 1: Graphical representation for $e_i \times^{\gamma} e_j$.

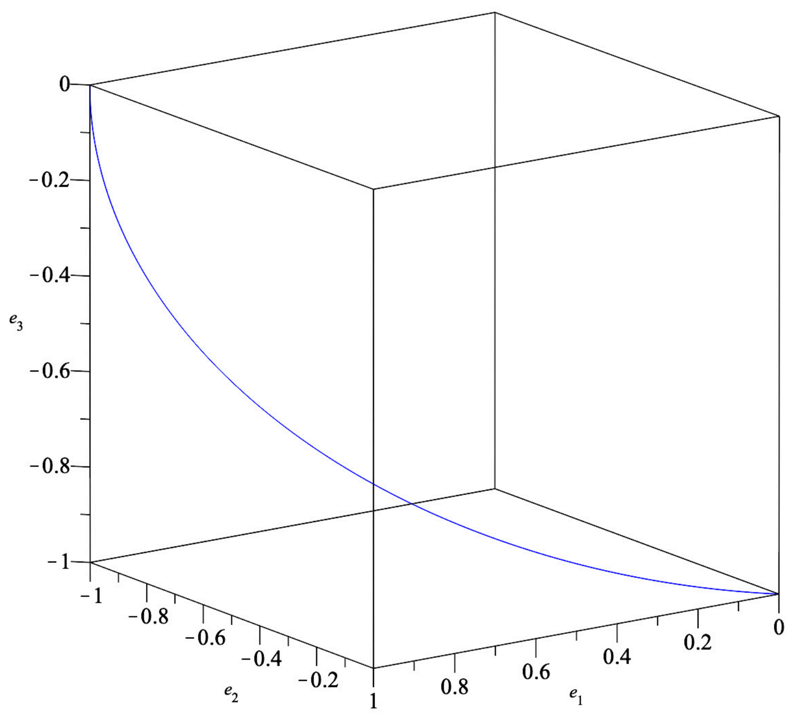

Figure 2: Graphical representation for $e_j \times^{\gamma} e_i$.

5. Standard Fractional Vector Cross Product for First Vector Pair

From above section we have understood that fractional cross product is the fractional rotation of angle $\frac{\gamma\pi}{2}$ of the vector on which the cross product operation is carried on and the rotation about the axis of the vector which is doing the cross product operation. The operation being linear we can use superposition to get the expression for the general Standard fractional vector cross product.

Let us take a vector $\bar{A} = (1)\hat{x}_k + (1)\hat{y}_k + (1)\hat{z}_k$

Using eqns (2) , (3) and (4) we have,

$$\begin{aligned} \hat{z}_k \times^\gamma \bar{A} &= \hat{z}_k \times^\gamma (1)\hat{x}_k + (1)\hat{y}_k + (1)\hat{z}_k \\ &= \cos\left(\frac{\gamma\pi}{2}\right)\hat{x}_k + \sin\left(\frac{\gamma\pi}{2}\right)\hat{y}_k - \cos\left(\frac{\gamma\pi}{2}\right)\hat{z}_k \\ &\quad + \cos\left(\frac{\gamma\pi}{2}\right)\hat{y}_k - \sin\left(\frac{\gamma\pi}{2}\right)\hat{x}_k - \cos\left(\frac{\gamma\pi}{2}\right)\hat{z}_k + 0 \end{aligned} \tag{5}$$

Hence,

$$\hat{z}_k \times^\gamma \bar{A} = \left(\cos\left(\frac{\gamma\pi}{2}\right) + \sin\left(\frac{\gamma\pi}{2}\right)\right)\hat{x}_k + \left(\cos\left(\frac{\gamma\pi}{2}\right) - \sin\left(\frac{\gamma\pi}{2}\right)\right)\hat{y}_k - 2\cos\left(\frac{\gamma\pi}{2}\right)\hat{z}_k \tag{6}$$

Alternatively, eqn (6) can be written as

$$\hat{z}_k \times^\gamma \bar{A} = \begin{bmatrix} \cos\left(\frac{\gamma\pi}{2}\right)\hat{x}_k & -\sin\left(\frac{\gamma\pi}{2}\right)\hat{x}_k & (0)\hat{x}_k \\ \sin\left(\frac{\gamma\pi}{2}\right)\hat{y}_k & \cos\left(\frac{\gamma\pi}{2}\right)\hat{y}_k & (0)\hat{y}_k \\ -\cos\left(\frac{\gamma\pi}{2}\right)\hat{z}_k & -\cos\left(\frac{\gamma\pi}{2}\right)\hat{z}_k & (0)\hat{z}_k \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{7}$$

Case 1: Putting $\gamma = 0$ in eqn (7) we have

$$\hat{z}_k \times^0 \bar{A} = \begin{bmatrix} 1\hat{x}_k & 0\hat{x}_k & (0)\hat{x}_k \\ 0\hat{y}_k & 1\hat{y}_k & (0)\hat{y}_k \\ -1\hat{z}_k & -1\hat{z}_k & (0)\hat{z}_k \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \hat{x}_k + \hat{y}_k - 2\hat{z}_k \tag{8}$$

Case 2: Putting $\gamma = 1$ in eqn (7) we have

$$\hat{z}_k \times \bar{A} = \begin{bmatrix} 0\hat{x}_k & -1\hat{x}_k & (0)\hat{x}_k \\ 1\hat{y}_k & 0\hat{y}_k & (0)\hat{y}_k \\ 0\hat{z}_k & 0\hat{z}_k & (0)\hat{z}_k \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -\hat{x}_k + \hat{y}_k \tag{9}$$

Alternatively, eqn (9) can be written as

$$\hat{z}_k \times \bar{A} = \begin{vmatrix} \hat{x}_k & \hat{y}_k & \hat{z}_k \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -\hat{x}_k + \hat{y}_k \tag{10}$$

Thus for $\gamma = 1$ fractional vector cross product works as a standard vector cross product.

Eqn (6) can be written as below

$$\hat{z}_k \times^{\gamma} \bar{A} = \left(\cos\left(\frac{\gamma\pi}{2}\right) + \sin\left(\frac{\gamma\pi}{2}\right)\right)\hat{x}_k + \left(\cos\left(\frac{\gamma\pi}{2}\right) - \sin\left(\frac{\gamma\pi}{2}\right)\right)\hat{y}_k + \delta_\gamma \cos\left(\frac{\gamma\pi}{2}\right)\hat{z}_k \tag{11}$$

where $\delta_\gamma = -2$ for $\gamma = 0$.

6. Standard Fractional Vector Cross product for Second Vector Pair

Now replacing $\bar{A} = (1,1,1)$ with $\bar{F}_k = (F_{xk}, F_{yk}, F_{zk}) = F_{xk}\hat{x}_k + F_{yk}\hat{y}_k + F_{zk}\hat{z}_k$ we have standard fractional vector cross product for $\gamma \neq 0$.

$$\hat{z}_k \times^{\gamma} \bar{F}_k = \begin{bmatrix} \cos\left(\frac{\gamma\pi}{2}\right)\hat{x}_k & -\sin\left(\frac{\gamma\pi}{2}\right)\hat{x}_k & (0)\hat{x}_k \\ \sin\left(\frac{\gamma\pi}{2}\right)\hat{y}_k & \cos\left(\frac{\gamma\pi}{2}\right)\hat{y}_k & (0)\hat{y}_k \\ -\cos\left(\frac{\gamma\pi}{2}\right)\hat{z}_k & -\cos\left(\frac{\gamma\pi}{2}\right)\hat{z}_k & (0)\hat{z}_k \end{bmatrix} \begin{bmatrix} F_{xk} \\ F_{yk} \\ F_{zk} \end{bmatrix} \tag{12}$$

Putting $\gamma = 1$ in above equation, we have

$$\hat{z}_k \times \bar{F}_k = \begin{bmatrix} 0\hat{x}_k & -1\hat{x}_k & (0)\hat{x}_k \\ 1\hat{y}_k & 0\hat{y}_k & (0)\hat{y}_k \\ 0\hat{z}_k & 0\hat{z}_k & (0)\hat{z}_k \end{bmatrix} \begin{bmatrix} F_{xk} \\ F_{yk} \\ F_{zk} \end{bmatrix} = -F_{yk}\hat{x}_k + F_{xk}\hat{y}_k + 0\hat{z}_k \tag{13}$$

which satisfies standard vector cross product.

$$\hat{z}_k \times \bar{F}_k = \begin{vmatrix} \hat{x}_k & \hat{y}_k & \hat{z}_k \\ 0 & 0 & 1 \\ F_{xk} & F_{yk} & F_{zk} \end{vmatrix} = -F_{yk}\hat{x}_k + F_{xk}\hat{y}_k + 0\hat{z}_k \tag{14}$$

Using eqns (12), (13) and (14) we can write

$$\hat{x}_k \times^{\gamma} \bar{F}_k = \begin{bmatrix} (0)\hat{x}_k & \cos\left(\frac{\gamma\pi}{2}\right)\hat{x}_k & \cos\left(\frac{\gamma\pi}{2}\right)\hat{x}_k \\ (0)\hat{y}_k & \cos\left(\frac{\gamma\pi}{2}\right)\hat{y}_k & -\sin\left(\frac{\gamma\pi}{2}\right)\hat{y}_k \\ (0)\hat{z}_k & \sin\left(\frac{\gamma\pi}{2}\right)\hat{z}_k & \cos\left(\frac{\gamma\pi}{2}\right)\hat{z}_k \end{bmatrix} \begin{bmatrix} F_{xk} \\ F_{yk} \\ F_{zk} \end{bmatrix} \tag{15}$$

Putting $\gamma = 1$ in above equation, we have

$$\hat{x}_k \times \bar{F}_k = \begin{bmatrix} (0)\hat{x}_k & 0\hat{x}_k & 0\hat{x}_k \\ (0)\hat{y}_k & 0\hat{y}_k & -1\hat{y}_k \\ (0)\hat{z}_k & 1\hat{z}_k & 0\hat{z}_k \end{bmatrix} \begin{bmatrix} F_{xk} \\ F_{yk} \\ F_{zk} \end{bmatrix} = 0\hat{x}_k - F_{zk}\hat{y}_k + F_{yk}\hat{z}_k \tag{16}$$

which satisfies standard vector cross product.

$$\hat{x}_k \times \bar{F}_k = \begin{vmatrix} \hat{x}_k & \hat{y}_k & \hat{z}_k \\ 1 & 0 & 0 \\ F_{xk} & F_{yk} & F_{zk} \end{vmatrix} = 0\hat{x}_k - F_{zk}\hat{y}_k + F_{yk}\hat{z}_k \tag{17}$$

Now we have

$$\hat{y}_k \times^{\gamma} \bar{F}_k = \begin{bmatrix} \cos\left(\frac{\gamma\pi}{2}\right)\hat{x}_k & 0\hat{x}_k & \sin\left(\frac{\gamma\pi}{2}\right)\hat{x}_k \\ -\cos\left(\frac{\gamma\pi}{2}\right)\hat{y}_k & 0\hat{y}_k & -\cos\left(\frac{\gamma\pi}{2}\right)\hat{y}_k \\ -\sin\left(\frac{\gamma\pi}{2}\right)\hat{z}_k & 0\hat{z}_k & \cos\left(\frac{\gamma\pi}{2}\right)\hat{z}_k \end{bmatrix} \begin{bmatrix} F_{xk} \\ F_{yk} \\ F_{zk} \end{bmatrix} \tag{18}$$

Putting $\gamma = 1$ in above equation, we have

$$\hat{y}_k \times \bar{F}_k = \begin{bmatrix} 0\hat{x}_k & 0\hat{x}_k & 1\hat{x}_k \\ 0\hat{y}_k & 0\hat{y}_k & 0\hat{y}_k \\ -1\hat{z}_k & 0\hat{z}_k & 0\hat{z}_k \end{bmatrix} \begin{bmatrix} F_{xk} \\ F_{yk} \\ F_{zk} \end{bmatrix} = F_{zk}\hat{x}_k + 0\hat{y}_k - F_{xk}\hat{z}_k \tag{19}$$

which satisfies standard vector cross product.

$$\hat{y}_k \times \bar{F}_k = \begin{vmatrix} \hat{x}_k & \hat{y}_k & \hat{z}_k \\ 0 & 1 & 0 \\ F_{xk} & F_{yk} & F_{zk} \end{vmatrix} = F_{zk}\hat{x}_k + 0\hat{y}_k - F_{xk}\hat{z}_k \tag{20}$$

From the eqns (12), (15) and (18) and superposing them together we get fractional cross product of vector $(\hat{x}_k, \hat{y}_k, \hat{z}_k)$ with the vector $\bar{F}_k = (F_{xk}, F_{yk}, F_{zk})$ for $\gamma \neq 0$ and $(\hat{x}_k, \hat{y}_k, \hat{z}_k) = (1)\hat{x}_k + (1)\hat{y}_k + (1)\hat{z}_k$, we have

$$\begin{aligned} (\hat{x}_k, \hat{y}_k, \hat{z}_k) \times^{\gamma} (F_{xk}, F_{yk}, F_{zk}) = & \left(F_{yk}\cos\left(\frac{\gamma\pi}{2}\right) + F_{zk}\cos\left(\frac{\gamma\pi}{2}\right)\right)\hat{x}_k \\ & + \left(F_{yk}\cos\left(\frac{\gamma\pi}{2}\right) - F_{zk}\sin\left(\frac{\gamma\pi}{2}\right)\right)\hat{y}_k \\ & + \left(F_{yk}\sin\left(\frac{\gamma\pi}{2}\right) + F_{zk}\cos\left(\frac{\gamma\pi}{2}\right)\right)\hat{z}_k \\ & + \left(F_{xk}\cos\left(\frac{\gamma\pi}{2}\right) + F_{zk}\sin\left(\frac{\gamma\pi}{2}\right)\right)\hat{x}_k \\ & + \left(-F_{xk}\cos\left(\frac{\gamma\pi}{2}\right) - F_{zk}\cos\left(\frac{\gamma\pi}{2}\right)\right)\hat{y}_k \\ & + \left(-F_{xk}\sin\left(\frac{\gamma\pi}{2}\right) + F_{zk}\cos\left(\frac{\gamma\pi}{2}\right)\right)\hat{z}_k \\ & + \left(F_{xk}\cos\left(\frac{\gamma\pi}{2}\right) - F_{yk}\sin\left(\frac{\gamma\pi}{2}\right)\right)\hat{x}_k \\ & + \left(F_{xk}\sin\left(\frac{\gamma\pi}{2}\right) + F_{yk}\cos\left(\frac{\gamma\pi}{2}\right)\right)\hat{y}_k \\ & + \left(-F_{xk}\cos\left(\frac{\gamma\pi}{2}\right) - F_{yk}\cos\left(\frac{\gamma\pi}{2}\right)\right)\hat{z}_k \end{aligned} \tag{21}$$

$$
\begin{aligned}
(\hat{x}_k,\hat{y}_k,\hat{z}_k)\times^{\gamma}(F_{xk},F_{yk},F_{zk}) = &\Bigg(2F_{xk}\cos\left(\frac{\gamma\pi}{2}\right)+F_{yk}\left(\cos\left(\frac{\gamma\pi}{2}\right)-\sin\left(\frac{\gamma\pi}{2}\right)\right)\\
&+F_{zk}\left(\cos\left(\frac{\gamma\pi}{2}\right)+\sin\left(\frac{\gamma\pi}{2}\right)\right)\Bigg)\hat{x}_k+F_{xk}\left(-\cos\left(\frac{\gamma\pi}{2}\right)+\sin\left(\frac{\gamma\pi}{2}\right)\right)\\
&+2F_{yk}\cos\left(\frac{\gamma\pi}{2}\right)+F_{zk}\left(-\cos\left(\frac{\gamma\pi}{2}\right)-\sin\left(\frac{\gamma\pi}{2}\right)\right)\hat{y}_k\\
&+F_{xk}\left(-\cos\left(\frac{\gamma\pi}{2}\right)-\sin\left(\frac{\gamma\pi}{2}\right)\right)\\
&+F_{yk}\left(-\cos\left(\frac{\gamma\pi}{2}\right)+\sin\left(\frac{\gamma\pi}{2}\right)\right)+2F_{zk}\cos\left(\frac{\gamma\pi}{2}\right)\hat{z}_k
\end{aligned}
\tag{22}
$$

Putting $\gamma=1$ we have

$$
(\hat{x}_k,\hat{y}_k,\hat{z}_k)\times(F_{xk},F_{yk},F_{zk}) = (-F_{yk}+F_{zk})\hat{x}_k+(F_{xk}-F_{zk})\hat{y}_k+(-F_{xk}+F_{yk})\hat{z}_k \tag{23}
$$

which satisfies standard vector cross product.

$$
\begin{aligned}
(\hat{x}_k,\hat{y}_k,\hat{z}_k)\times(F_{xk},F_{yk},F_{zk}) &= \begin{vmatrix}\hat{x}_k & \hat{y}_k & \hat{z}_k\\ 1 & 1 & 1\\ F_{xk} & F_{yk} & F_{zk}\end{vmatrix}\\
&= (-F_{yk}+F_{zk})\hat{x}_k+(F_{xk}-F_{zk})\hat{y}_k+(-F_{xk}+F_{yk})\hat{z}_k
\end{aligned}
\tag{24}
$$

This is similar to standard vector cross product.

7. Fractional Curl with Standard Fractional Vector Cross Product

Let us have a vector field which has variation in $\hat{z}$ direction. This can be transformed as a vector field $\bar{F}_k(z)=F_x\hat{x}_k+F_y\hat{y}_k+F_z\hat{z}_k$ in k domain. We have to find gradient in z direction only, since the vector field has the variation in that direction only. We simply multiply $(ik)^{\alpha}$ which is given in eqn (7).

$$
ik\hat{z}_k\times^{\gamma}\bar{F}_k = \begin{bmatrix}
\cos\left(\frac{\gamma\pi}{2}\right)\hat{x}_k(ik)^{\gamma} & -\sin\left(\frac{\gamma\pi}{2}\right)\hat{x}_k(ik)^{\gamma} & (0)\hat{x}_k(ik)^{\gamma}\\
\sin\left(\frac{\gamma\pi}{2}\right)\hat{y}_k(ik)^{\gamma} & \cos\left(\frac{\gamma\pi}{2}\right)\hat{y}_k(ik)^{\gamma} & (0)\hat{y}_k(ik)^{\gamma}\\
-\cos\left(\frac{\gamma\pi}{2}\right)\hat{z}_k(ik)^{\gamma} & -\cos\left(\frac{\gamma\pi}{2}\right)\hat{z}_k(ik)^{\gamma} & (0)\hat{z}_k(ik)^{\gamma}
\end{bmatrix}
\begin{bmatrix}F_{xk}\\ F_{yk}\\ F_{zk}\end{bmatrix}
\tag{25}
$$

Fourier inversion of eqn (25) gives fractional curl at $\gamma\neq 0$

$$
\nabla_z\times^{\gamma}\bar{F} = \begin{bmatrix}
\cos\left(\frac{\gamma\pi}{2}\right)\hat{x}({}_{-\infty}D_z^{\gamma}) & -\sin\left(\frac{\gamma\pi}{2}\right)\hat{x}({}_{-\infty}D_z^{\gamma}) & (0)\hat{x}({}_{-\infty}D_z^{\gamma})\\
\sin\left(\frac{\gamma\pi}{2}\right)\hat{y}({}_{-\infty}D_z^{\gamma}) & \cos\left(\frac{\gamma\pi}{2}\right)\hat{y}({}_{-\infty}D_z^{\gamma}) & (0)\hat{y}({}_{-\infty}D_z^{\gamma})\\
-\cos\left(\frac{\gamma\pi}{2}\right)\hat{z}({}_{-\infty}D_z^{\gamma}) & -\cos\left(\frac{\gamma\pi}{2}\right)\hat{z}({}_{-\infty}D_z^{\gamma}) & (0)\hat{z}(({}_{-\infty}D_z^{\gamma})
\end{bmatrix}
\begin{bmatrix}f_x\\ f_y\\ f_z\end{bmatrix}
\tag{26}
$$

Putting $\gamma = 1$ in eqn (26) we have

$$\nabla_z \times \bar{F} = \begin{bmatrix} 0 & -\hat{x}({}_{-\infty}D_z^1) & 0 \\ \hat{y}({}_{-\infty}D_z^1) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = -\hat{x}({}_{-\infty}D_z^1)f_y + \hat{y}({}_{-\infty}D_z^1)f_x \tag{27}$$

which is alternatively equivalent to

$$\nabla_z \times \bar{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & {}_{-\infty}D_z^1 \\ f_x & f_y & f_z \end{vmatrix} = -\hat{x}({}_{-\infty}D_z^1)f_y + \hat{y}({}_{-\infty}D_z^1)f_x \tag{28}$$

Similarly we have

$$ik\hat{x}_k \times^{\gamma} \bar{F}_k = \begin{bmatrix} (0)\hat{x}_k(ik)^{\gamma} & \cos\left(\frac{\gamma\pi}{2}\right)\hat{x}_k(ik)^{\gamma} & \cos\left(\frac{\gamma\pi}{2}\right)\hat{x}_k(ik)^{\gamma} \\ (0)\hat{y}_k(ik)^{\gamma} & \cos\left(\frac{\gamma\pi}{2}\right)\hat{y}_k(ik)^{\gamma} & -\sin\left(\frac{\gamma\pi}{2}\right)\hat{y}_k(ik)^{\gamma} \\ (0)\hat{z}_k(ik)^{\gamma} & \sin\left(\frac{\gamma\pi}{2}\right)\hat{z}_k(ik)^{\gamma} & \cos\left(\frac{\gamma\pi}{2}\right)\hat{z}_k(ik)^{\gamma} \end{bmatrix} \begin{bmatrix} F_{xk} \\ F_{yk} \\ F_{zk} \end{bmatrix} \tag{29}$$

Fourier inversion of eqn (29) gives fractional curl at $\gamma \neq 0$

$$\nabla_x \times^{\gamma} \bar{F} = \begin{bmatrix} (0)\hat{x}({}_{-\infty}D_x^{\gamma}) & \cos\left(\frac{\gamma\pi}{2}\right)\hat{x}({}_{-\infty}D_x^{\gamma}) & \cos\left(\frac{\gamma\pi}{2}\right)\hat{x}({}_{-\infty}D_x^{\gamma}) \\ (0)\hat{y}({}_{-\infty}D_x^{\gamma}) & \cos\left(\frac{\gamma\pi}{2}\right)\hat{y}({}_{-\infty}D_x^{\gamma}) & -\sin\left(\frac{\gamma\pi}{2}\right)\hat{y}({}_{-\infty}D_x^{\gamma}) \\ (0)\hat{z}({}_{-\infty}D_x^{\gamma}) & \sin\left(\frac{\gamma\pi}{2}\right)\hat{z}({}_{-\infty}D_x^{\gamma}) & \cos\left(\frac{\gamma\pi}{2}\right)\hat{z}({}_{-\infty}D_x^{\gamma}) \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \tag{30}$$

Putting $\gamma = 1$ in eqn (30) we have

$$\nabla_x \times \bar{F} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -({}_{-\infty}D_x^1)\hat{y} \\ 0 & ({}_{-\infty}D_x^{\gamma})\hat{z} & 0 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = -({}_{-\infty}D_x^1)f_z\hat{y} + ({}_{-\infty}D_x^1)f_y\hat{z} \tag{31}$$

which is alternatively equivalent to

$$\nabla_x \times \bar{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ {}_{-\infty}D_x^1 & 0 & 0 \\ f_x & f_y & f_z \end{vmatrix} = -({}_{-\infty}D_x^1)f_z\hat{y} + ({}_{-\infty}D_x^1)f_y\hat{z} \tag{32}$$

Again we have

$$ik\hat{y}_k \times^{\gamma} \bar{F}_k = \begin{bmatrix} \cos\left(\frac{\gamma\pi}{2}\right)\hat{x}_k(ik)^{\gamma} & 0\hat{x}_k(ik)^{\gamma} & \sin\left(\frac{\gamma\pi}{2}\right)\hat{x}_k(ik)^{\gamma} \\ -\cos\left(\frac{\gamma\pi}{2}\right)\hat{y}_k(ik)^{\gamma} & 0\hat{y}_k(ik)^{\gamma} & -\cos\left(\frac{\gamma\pi}{2}\right)\hat{y}_k(ik)^{\gamma} \\ -\sin\left(\frac{\gamma\pi}{2}\right)\hat{z}_k(ik)^{\gamma} & 0\hat{z}_k(ik)^{\gamma} & \cos\left(\frac{\gamma\pi}{2}\right)\hat{z}_k(ik)^{\gamma} \end{bmatrix} \begin{bmatrix} F_{xk} \\ F_{yk} \\ F_{zk} \end{bmatrix} \tag{33}$$

Fourier inversion of eqn (33) gives fractional curl at $\gamma \neq 0$

$$\nabla_y \times^{\gamma} \bar{F} = \begin{bmatrix} \cos\left(\frac{\gamma\pi}{2}\right)\hat{x}({}_{-\infty}D_y^{\gamma}) & 0\hat{x}({}_{-\infty}D_y^{\gamma}) & \sin\left(\frac{\gamma\pi}{2}\right)\hat{x}({}_{-\infty}D_y^{\gamma}) \\ -\cos\left(\frac{\gamma\pi}{2}\right)\hat{y}({}_{-\infty}D_y^{\gamma}) & 0\hat{y}({}_{-\infty}D_y^{\gamma}) & -\cos\left(\frac{\gamma\pi}{2}\right)\hat{y}({}_{-\infty}D_y^{\gamma}) \\ -\sin\left(\frac{\gamma\pi}{2}\right)\hat{z}({}_{-\infty}D_y^{\gamma}) & 0\hat{z}({}_{-\infty}D_y^{\gamma}) & \cos\left(\frac{\gamma\pi}{2}\right)\hat{z}({}_{-\infty}D_y^{\gamma}) \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \tag{34}$$

Putting $\gamma = 1$ in eqn (34) we have

$$\begin{aligned} \nabla_y \times \bar{F} &= \begin{bmatrix} 0 & 0 & ({}_{-\infty}D_y^1)\hat{x} \\ 0 & 0 & 0 \\ -({}_{-\infty}D_y^1)\hat{z} & 0 & 0 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \\ &= ({}_{-\infty}D_y^{\gamma})\hat{x}f_z - ({}_{-\infty}D_y^{\gamma})\hat{z}f_x \end{aligned} \tag{35}$$

which is alternatively equivalent to

$$\nabla_y \times \bar{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & {}_{-\infty}D_y^1 & 0 \\ f_x & f_y & f_z \end{vmatrix} = -({}_{-\infty}D_y^1)f_z\hat{y} + ({}_{-\infty}D_y^1)f_y\hat{z} \tag{36}$$

Adding (26), (30) and (34) which are components of $\nabla \times^{\gamma} \bar{F}$ where $\nabla = \hat{x}\nabla_x + \hat{y}\nabla_y + \hat{z}\nabla_z$ and $F = \hat{x}f_x + \hat{y}f_y + \hat{z}f_z$ for $\gamma \neq 0$ we have

$$\begin{aligned} \nabla \times^{\gamma} \bar{F} = \hat{x}\Bigg\{ & \cos\left(\frac{\gamma\pi}{2}\right)\{({}_{-\infty}D_y^{\gamma}) + ({}_{-\infty}D_z^{\gamma})\}f_x + \left\{-\cos\left(\frac{\gamma\pi}{2}\right)({}_{-\infty}D_x^{\gamma}) - \sin\left(\frac{\gamma\pi}{2}\right)({}_{-\infty}D_z^{\gamma})\right\}f_y \\ & - \left\{\cos\left(\frac{\gamma\pi}{2}\right)({}_{-\infty}D_x^{\gamma}) - \sin\left(\frac{\gamma\pi}{2}\right)({}_{-\infty}D_y^{\gamma})\right\}f_z\Bigg\} + \hat{y}\left\{\left\{-\cos\left(\frac{\gamma\pi}{2}\right)({}_{-\infty}D_y^{\gamma}) + \sin\left(\frac{\gamma\pi}{2}\right)\hat{y}({}_{-\infty}D_z^{\gamma})\right\}f_x\right. \\ & + \left\{\cos\left(\frac{\gamma\pi}{2}\right)\{({}_{-\infty}D_x^{\gamma}) + ({}_{-\infty}D_z^{\gamma})\}\right\}f_y + \left.\left\{-\sin\left(\frac{\gamma\pi}{2}\right)({}_{-\infty}D_x^{\gamma}) - \cos\left(\frac{\gamma\pi}{2}\right)({}_{-\infty}D_y^{\gamma})\right\}f_z\right\} \\ & + \hat{z}\left\{-\sin\left(\frac{\gamma\pi}{2}\right)({}_{-\infty}D_y^{\gamma}) - \cos\left(\frac{\gamma\pi}{2}\right)({}_{-\infty}D_z^{\gamma})\right\}f_x + \left\{\sin\left(\frac{\gamma\pi}{2}\right)({}_{-\infty}D_x^{\gamma}) - \cos\left(\frac{\gamma\pi}{2}\right)({}_{-\infty}D_z^{\gamma})\right\}f_y \\ & + \left\{\cos\left(\frac{\gamma\pi}{2}\right)\{({}_{-\infty}D_x^{\gamma}) + ({}_{-\infty}D_y^{\gamma})\}f_z\right\} \end{aligned} \tag{37}$$

$$\nabla \times^{\gamma} \bar{F} = \hat{x}\left[\left(\frac{\partial^{\gamma} f_x}{\partial y^{\gamma}} + \frac{\partial^{\gamma} f_x}{\partial z^{\gamma}} - \frac{\partial^{\gamma} f_y}{\partial x^{\gamma}} - \frac{\partial^{\gamma} f_z}{\partial x^{\gamma}}\right)\cos\left(\frac{\gamma\pi}{2}\right) + \left(\frac{\partial^{\gamma} f_z}{\partial y^{\gamma}} - \frac{\partial^{\gamma} f_y}{\partial z^{\gamma}}\right)\sin\left(\frac{\gamma\pi}{2}\right)\right]$$
$$+\hat{y}\left[\left(\frac{\partial^{\gamma} f_y}{\partial z^{\gamma}} + \frac{\partial^{\gamma} f_y}{\partial x^{\gamma}} - \frac{\partial^{\gamma} f_x}{\partial y^{\gamma}} - \frac{\partial^{\gamma} f_z}{\partial y^{\gamma}}\right)\cos\left(\frac{\gamma\pi}{2}\right) + \left(\frac{\partial^{\gamma} f_x}{\partial z^{\gamma}} - \frac{\partial^{\gamma} f_z}{\partial x^{\gamma}}\right)\sin\left(\frac{\gamma\pi}{2}\right)\right] \quad (38)$$
$$+\hat{z}\left[\left(\frac{\partial^{\gamma} f_z}{\partial x^{\gamma}} + \frac{\partial^{\gamma} f_z}{\partial y^{\gamma}} - \frac{\partial^{\gamma} f_x}{\partial z^{\gamma}} - \frac{\partial^{\gamma} f_y}{\partial z^{\gamma}}\right)\cos\left(\frac{\gamma\pi}{2}\right) + \left(\frac{\partial^{\gamma} f_y}{\partial x^{\gamma}} - \frac{\partial^{\gamma} f_x}{\partial y^{\gamma}}\right)\sin\left(\frac{\gamma\pi}{2}\right)\right]$$

Putting $\gamma = 1$ in eqn (38) we have

$$\nabla \times \bar{F} = \hat{x}\left[\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\right] + \hat{y}\left[\left(\frac{\partial f_x}{\partial z} - \frac{\partial \alpha f_z}{\partial x}\right)\right] + \hat{z}\left[\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\right]$$
$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial^1}{\partial x^1} & \frac{\partial^1}{\partial y^1} & \frac{\partial^1}{\partial z^1} \\ f_x & f_y & f_z \end{vmatrix} \quad (39)$$

From eqn (38) and (39) we have fractional curl defined as:

$$\nabla \times^{\gamma} \bar{F} = \cos\left(\frac{\gamma\pi}{2}\right)\begin{bmatrix} \frac{\partial^{\gamma}}{\partial y^{\gamma}} + \frac{\partial^{\gamma}}{\partial z^{\gamma}} & -\frac{\partial^{\gamma}}{\partial x^{\gamma}} & -\frac{\partial^{\gamma}}{\partial x^{\gamma}} \\ -\frac{\partial^{\gamma}}{\partial y^{\gamma}} & \frac{\partial^{\gamma}}{\partial x^{\gamma}} + \frac{\partial^{\gamma}}{\partial z^{\gamma}} & -\frac{\partial^{\gamma}}{\partial y^{\gamma}} \\ -\frac{\partial^{\gamma}}{\partial z^{\gamma}} & -\frac{\partial^{\gamma}}{\partial z^{\gamma}} & \frac{\partial^{\gamma}}{\partial x^{\gamma}} + \frac{\partial^{\gamma}}{\partial y^{\gamma}} \end{bmatrix}\begin{bmatrix} \hat{x}f_x \\ \hat{y}f_y \\ \hat{z}f_z \end{bmatrix}$$
$$+\sin\left(\frac{\gamma\pi}{2}\right)\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial^{\gamma}}{\partial x^{\gamma}} & \frac{\partial^{\gamma}}{\partial y^{\gamma}} & \frac{\partial^{\gamma}}{\partial z^{\gamma}} \\ f_x & f_y & f_z \end{vmatrix} \quad (40)$$

$$\nabla \times^{\gamma} \bar{F} = \cos\left(\frac{\gamma\pi}{2}\right)\begin{bmatrix} \nabla^{\gamma}_{yz} & -\partial^{\gamma}_x & -\partial^{\gamma}_x \\ -\partial^{\gamma}_y & \nabla^{\gamma}_{zx} & -\partial^{\gamma}_y \\ -\partial^{\gamma}_z & -\partial^{\gamma}_z & \nabla^{\gamma}_{xy} \end{bmatrix}\begin{bmatrix} \hat{x}f_x \\ \hat{y}f_y \\ \hat{z}f_z \end{bmatrix} + \sin\left(\frac{\gamma\pi}{2}\right)\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial^{\gamma}_x & \partial^{\gamma}_y & \partial^{\gamma}_z \\ f_x & f_y & f_z \end{vmatrix} \quad (41)$$

where $\nabla^{\gamma}_{yz} = \frac{\partial^{\gamma}}{\partial y^{\gamma}} + \frac{\partial^{\gamma}}{\partial z^{\gamma}}, \nabla^{\gamma}_{zx} = \frac{\partial^{\gamma}}{\partial z^{\gamma}} + \frac{\partial^{\gamma}}{\partial x^{\gamma}}, \nabla^{\gamma}_{xy} = \frac{\partial^{\gamma}}{\partial x^{\gamma}} + \frac{\partial^{\gamma}}{\partial y^{\gamma}}, \frac{\partial^{\gamma}}{\partial x^{\gamma}} = \partial^{\gamma}_x, \frac{\partial^{\gamma}}{\partial y^{\gamma}} = \partial^{\gamma}_y, \frac{\partial^{\gamma}}{\partial z^{\gamma}} = \partial^{\gamma}_z$

8. Conclusion

This research focuses on establishing certain properties of SFVCP with vector pair. This new definition satisfies all the properties of geometrical reality and the standard vector cross product result at $\gamma = 1$ which makes it more applicable in real sense. As this definition is new it will open path for many researches in various fields of electromagnetic theory, electrodynamics, elastodynamics, fluid flow etc.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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