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Controllability of second order neutral impulsive fuzzy functional differential equations with Non-Local conditions

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Abstract

In this paper, the controllability of fuzzy solutions for a second-order nonlocal impulsive neutral functional differential equation with both nonlocal and impulsive conditions in terms of fuzzy are considered. The sufficient condition of controllability is developed using the Banach fixed point theorem and a fuzzy number whose values are normal, convex, upper semi-continuous, and compactly supported fuzzy sets with the Hausdorff distance between α-cuts at its maximum. The α-cut approaches allow to translate a system of fuzzy differential equations into a system of ordinary differential equations to the endpoints of the states. An example of the application is given at the end to demonstrate the results. These kinds of systems come in use for designing landing systems for planes and spacecraft, as well as car suspension systems.

Keywords: Controllability, Neutral Impulsive Differential Equation, Banach Fixed Point Theorem, Fuzzy solution, Non-local initial conditions.

AMS Subject Classification 2020: 03E72, 93B05, 47H10, 34K40, 34K45.

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1. Introduction

Control theory is a fascinating part of application-oriented mathematics that deals with the fundamental ideas that support control framework analysis and design. The main objective of the control theory is to perform specific tasks by the system applying appropriate control. Controllability is well recognized in the context of control systems and comprises a central location. In controllability, one analyses the possibility of changing a system from a given state (initial state) to a certain required final state by using a set of permissible controls. One main presumption in the control system is that all its components are involved with complete precision. Moreover, control systems related to reasonable circumstances are characterized by fuzziness. Fuzzy set theory, introduced by [21] is competent to take care of such kind of fuzziness. Since a fuzzy differential equation describes a fuzzy control system with some initial conditions (fuzzy or non-fuzzy). So, first, we study some results pertaining to fuzzy differential equations, see ([10], [16], [11]), and references therein.

Neutral functional differential equations emerge in various disciplines of applied mathematics and so these equations have become more important in a few decades. For more detail on neutral functional differential equations, refer [13], and the references therein. Different kinds of mathematical models in the study of population dynamics, biology, ecology, and epidemics can be represented as impulsive neutral differential equations. The theory of these equations has been examined by many authors ([9], [14], [20]). The vehicle industry has closely examined and is still curious about, the vehicle suspension system since it is the component that physically isolates the vehicle body from the wheels of the car to move forward the ride stability, comfort, and street dealing with of vehicles.

The issues which can be modeled in form of impulsive control systems experience sudden changes at certain focuses of time. Generally, impulses are not defined in a precise manner. So fuzzy impulsive condition may be better than to simple impulsive condition. For more details on fuzzy and non-fuzzy impulsive differential equations, we refer to see ([5], [18]) and references therein. [19] studied the periodic boundary value problems for second-order impulsive integrodifferential equations.

[7] studied the controllability for the following impulsive fuzzy neutral functional integrodifferential equations using Banach fixed point theorem.

$$\left.\begin{aligned} \frac{d}{dt}[x(t)+g(t,x_t)] &= Ax(t)+f(t,x_t,\int_0^{\infty}q(t,s,x_s)ds)+u(t);t\in J=[0,T],\\ x(0) &= \varphi\in E^n,\\ \Delta x(t_k) &= I_k x(t_k),t\neq t_k,k=1,2,\dots,m \end{aligned}\right\} \tag{1.1}$$

The controllability of impulsive second-order semilinear fuzzy integrodifferential control systems with nonlocal initial conditions has been studied by [17]. Controllability of second-order impulsive neutral integrodifferential systems with an infinite delay has been studied by [3].

Recently, [1] studied the controllability results of fuzzy solutions for the following first-order nonlocal impulsive neutral functional differential equation using the Banach fixed point theorem

$$\left.\begin{aligned} \frac{d}{dt}[x(t)-h(t,x_t)] &= Ax(t)+f(t,x_t)+u(t);t\in J=[0,T],\\ \Delta x(t_k) &= I_k x(t_k^-),t\neq t_k,k=1,2,\dots,p\\ x(t)+g(x_{\tau_1},x_{\tau_2},\dots,x_{\tau_p})(t) &= \varphi(t),t\in[-r,0] \end{aligned}\right\} \tag{1.2}$$

[6] studied the controllability of the following second-order neutral impulsive differential inclusions with non-local conditions

$$\left.\begin{aligned} \frac{d}{dt}[x'(t)-F(t,x(h_1(t))] &\in Ax(t)+G(t,x(h_2(t)),x'(h_2(t)))+Bu(t);\\ \Delta x(t_k) &= I_k x(t_k^-),t\neq t_k,k=1,2,\dots,p\\ \Delta x'(t_k) &= I_k^- x(t_k^-),t\neq t_k,k=1,2,\dots,p\\ x(0)+g(x) &= x_0;x'(0)=y_0,t\in J=[0,b];t\neq t_k. \end{aligned}\right\} \tag{1.3}$$

[8] studied the existence of fuzzy solutions for nonlocal impulsive neutral functional differential equations. In this paper, we attempt to establish controllability results for a class of fuzzy control systems governed by a fuzzy differential equation of second order coupled with nonlocal and impulsive conditions using the α-cut technique. In fact, nonlocal conditions are more viable for depicting the physical measurement rather than classical conditions (see for instance ([12], [4]), and references therein). We have discussed the controllability of fuzzy solutions for the following second-order non-local functional differential equations with an impulse which is an extension to the work done in [15]. Here both non-local as well as impulsive conditions are considered fuzzy.

$$\left.\begin{aligned} \frac{d}{dt}[x'(t)-h(t,x_t)] &= Ax(t)+f(t,x_t,x_{t'})+Bu(t); t\in J=[0,T],\\ \Delta x(t_k) &= I_k x(t_k^-), t\neq t_k, k=1,2,\cdots,p\\ \Delta x'(t_k) &= I_k^- x(t_k^-), t\neq t_k, k=1,2,\cdots,p\\ x(0)+g(x_{\tau_1},x_{\tau_2},\cdots,x_{\tau_p})(t) &= \varphi(t), t\in[-r,0],\\ x'(0) &= y_0 \end{aligned}\right\} \tag{1.4}$$

where $A,B:J\to E^n$ is the fuzzy coefficient, E^n is the set of all upper semi-continuous, convex, normal fuzzy numbers with bounded α levels.

The functions $f: J\times C([-r,0],E^n)\times C([-r,0],E^n)\to E^n$, $h: J\times C([-r,0],E^n)\to E^n$ and $g:(C[-r,0],E^n)^p \to E^n$ are non-linear regular fuzzy functions and $\varphi:[-r,0]\to E^n, 0<t_1<t_2<\ldots<t_p\le T$, $p\in\mathbb{N}$. $u:J\to E^n$ is an admissible control function and $I_k, I_k^-\in C(E^n,E^n)$ are bounded functions.

$\Delta x(t_k)=x(t_k^+)-x(t_k^-), \Delta x'(t_k)=x'(t_k^+)-x'(t_k^-), x(t_k^-), x'(t_k^-), x(t_k^+)$ and $x'(t_k^+)$ represents the left and right limits of $x(t)$ and $x'(t)$ at $t=t_k, k=1,2,\cdots,p$; respectively,

$x(t_k^+)=\lim_{h\to0^+}x(t_k+h)$, $x'(t_k^+)=\lim_{h\to0^+}x'(t_k+h)$, $x(t_k^-)=\lim_{h\to0^+}x(t_k-h)$ and $x'(t_k^-)=\lim_{h\to0^+}x'(t_k-h)$.

Moreover, $x_t(.)$ represents the history where $x_t(\theta)=x(t+\theta);\theta\in[-r,0]$.

Let Ω be the space given by $\Omega=\{x \mid x:J\to E^n; x_k\in C(J,E^n): k=1,2,\ldots,p$ and there exists $x(t_k^-)$ and $x(t_k^+); k=1,2,\ldots,p$ with

$$x(t_k^-)=x(t_k), x(0)+g(x_{\tau_1},x_{\tau_2},\ldots,x_{\tau_p})(t)=\varphi(t)\}.$$

We define $\Omega'=\Omega\cap C(J,E^n)=\{x' \mid x':J\to E^n; x_k'\in C(J,E^n): k=1,2,\ldots,p$ and there exists $x'(t_k^-)$ and $x'(t_k^+); k=1,2,\cdots,p$ with $x'(t_k^-)=x(t_k)$ and $x'(0)=y_0\}$.

We also present phase space $\mathcal{B}_h=\{\varphi:[-\infty,0]\to E$ such that for any $r>0, \varphi(0)$ is bounded and measurable function on $[-r,0]$ and $\int_{-\infty}^{0} h(s)\sup_{s<\theta\le0}|\varphi(\theta)|ds<+\infty\}$, where $\mathcal{B}_h$ is endowed with the norm $\|\varphi\|_{\mathcal{B}_h}=\int_{-\infty}^{0}h(s)\sup_{s<\theta\le0}|\varphi(\theta)|ds, \forall\varphi\in\mathcal{B}_h$.

Note that, $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space.

The paper is organized as follows: Section 2 summarizes the fundamental heuristics. The controllability results of the fuzzy solutions to non-local second-order neutral functional differential equations with impulse are proved using the Banach fixed point theorem in Section 3. An example has been provided to support the theory in Section 4. Section 5 contains an application with a graphical representation of the solution and finally, the conclusion is given in Section 6.

2. Fundamental Heuristics

2.1 Definition: Fuzzy Set

A fuzzy set $A\subseteq X\neq\varphi$ is characterized by its membership function $A:X\to[0,1]$ and $A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x\in X$.

2.2 Definition

Let $CC(\mathcal{R}^n)$ denote the family of all nonempty, compact, and convex subsets of $\mathcal{R}^n$. Define addition and scalar multiplication in $CC(\mathcal{R}^n)$ by

$$A + B = \{z : z = x + y, x \in A, y \in B\}$$

and

$$\lambda A = \{z : z = \lambda x, x \in A\} \ \forall \lambda \geq 0 \text{ and } \forall A, B \in CC(\mathcal{R}^n).$$

Let $J = [a,b] \subset \mathcal{R}$ be a compact interval and denote
$E^n = \{w : \mathcal{R}^n \to [0,1]$ such that w satisfies $(1) - (4)$ as below$\}$:

1. w is normal, that is, there exists an $\hat{x}_0 \in \mathcal{R}^n$ such that $w(\hat{x}_0) = 1$.
2. w is fuzzy convex, that is, $w(\lambda \hat{x} + (1-\lambda)\hat{z}) \geq min(w(\hat{x}), w(\hat{z}))$.
3. w is upper semi-continuous at $\hat{x}_0$, that is, $w(\hat{x}_0) \geq \overline{\lim}_{k\to\infty} w(\hat{x}_k)$ for any $\hat{x}_k \in \mathcal{R}^n, (k = 0,1,2,...), \hat{x}_k \to \hat{x}_0$.
4. $[w]^0 = \overline{(\hat{x} \in \mathcal{R}^n : w(\hat{x}) > 0)}$ is compact.

For, $0 < \alpha \leq 1$ we denote $[w]^\alpha = \{x \in \mathcal{R}^n : w(\hat{x}) \geq \alpha\}$.

Then from (1)–(4), it follows that the α-level sets $[w]^\alpha \in CC(\mathcal{R}^n)$. If $g : \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$ is a function, then by using Zadeh's extension principle, we can extend g to $E^n \times E^n \to E^n$ by the equation

$$[g(w,v)(z)] = \sup_{z=g(x,y)} min\{w(x), v(y)\}.$$

It is already known that $[g(w,v)]^\alpha = g([w]^\alpha, [v]^\alpha) \ \forall w,v \in E^n, 0 \leq \alpha \leq 1$ and g is a continuous function.

Further, we have $[w+v]^\alpha = [w]^\alpha + [v]^\alpha, [kw]^\alpha = k[w]^\alpha$ where $w,v \in E^n, k \in \mathcal{R}, 0 \leq \alpha < 1$.

Let A and B be two non-empty bounded subsets of $\mathcal{R}^n$. The Hausdorff metric defines the distance between A and B

$$H_d(A,B) = max\{\sup_{a\in A}\inf_{b\in B} \| a - b \|, \sup_{b\in B}\inf_{a\in A} \| a - b \|\},$$

where $\|\cdot\|$ denotes the usual Euclidean norm in $\mathcal{R}^n$. Then, $(CC(\mathcal{R}^n), H_d)$ is a complete and separable metric space [7].

2.3 Definition

The complete metric d_∞ on E^n is defined by

$$d_\infty(w,v) = \sup_{0<\alpha\leq 1} H_d([w]^\alpha, [v]^\alpha) = \sup_{0<\alpha\leq 1} | w_l^\alpha - v_l^\alpha, w_r^\alpha - v_r^\alpha |$$

for any $w,v,z \in E^n$, which satisfies $H_d(w+z, v+z) = H_d(w,v)$.

Hence, (E^n, d_∞) is a complete metric space [22].

2.4 Definition

The supremum metric H_1 on $C(J, E^n)$ is defined by

$$H_1(w,v) = \sup_{0\leq t\leq T} d_\infty(w(t), v(t))$$

Hence, $(C(J,E^n), H_1)$ is a complete metric space [24].

2.5 Definition

The derivative $x'(t)$ of a fuzzy process $x \in E^n$ is defined by

$$[x'(t)]^\alpha = [(x_l^\alpha)'(t), (x_r^\alpha)'(t)]$$

provided that the equation defines a fuzzy set $x'(t) \in E^n$ [24].

2.6 Definition

The fuzzy integral $\int_a^b x(t)dt$, $a, b \in [0, T]$ is defined by

$$\left[\int_a^b x(t)dt\right]^\alpha = \left[\int_a^b (x_l^\alpha)(t), \int_a^b (x_r^\alpha)(t)\right]$$

provided that the Lebesgue integrals on the right-hand side exist [24].

2.7 Definition

A mapping $f : J \to E^n$ is strongly measurable if, the set valued map $f_\alpha : J \to CC(\mathfrak{R}^n)$ defined by $f_\alpha(t) = [f(t)]^\alpha$ is Lebesgue measurable when $CC(\mathfrak{R}^n)$ has the topology induced by the Hausdorff metric [23].

2.8 Definition

A mapping $f : J \times E^n \to E^n$ is called level wise continuous at a point $(t_0, x_0) \in J \times E^n$ provided, for any fixed $\alpha \in [0,1]$ and arbitrary $\varepsilon > 0$, there exists a $\delta(\varepsilon, \alpha) > 0$, such that $H_d([f(t,x)]^\alpha, [f(t_0, x_0)]^\alpha < \varepsilon)$ whenever $|t - t_0| < \delta(\varepsilon, \alpha)$ and $H_d([x]^\alpha, [x_0]^\alpha) < \delta(\varepsilon, \alpha), \forall t \in J, x \in E^n$ [23].

2.9 Definition

A mapping $f : J \to E^n$ is called level wise continuous at $t_0 \in J$ if the multivalued map $f_\alpha(t) = [f(t)]^\alpha$ is continuous at $t = t_0$ with respect to the Hausdorff metric for all $\alpha \in [0,1]$ [23].

A map $f : J \to E^n$ is said to be integrably bounded if there is an integrable function $h(t)$, such that $\| x(t) | \le h(t)$ for every $x(t) \in f_0(t)$.

2.10 Definition

A strongly measurable and integrably bounded map $f : J \to E^n$ is considered integrable over J, if $\int_0^T f(t)dt \in E^n$. If $f : J \to E^n$ is strongly measurable and integrably bounded, then f is integrable [23].

2.11 Definition

A system is said to be controllable in E^n, if there exists an admissible control function $u(t)$ using which it is possible to steer a system from any arbitrary state to desired final state.

2.12 Definition

The α-cut or α-level set of A is denoted by A^α or $[A]_\alpha$ and is defined as $A^\alpha = \{x \in E^n : A(x) \ge \alpha\}$ for $\alpha \in (0,1]$.

3. Controllability Results

3.1 Assumptions

Assume the following hypothesis.

H1. [16] C(t) is a fuzzy number satisfying $\frac{dC(t)y}{dx} \in C^1(J;E^n) \cap C(J;E^n) \forall y \in E^n$. Since fuzzy integral is a fuzzy number so there exists a fuzzy number $S(t) \in E^n$ such as

$$S(t)y = \int_0^t C(s)yds \text{ ,with satisfying } \frac{dS(t)y}{dx} \in C^1(J;E^n) \cap C(J;E^n).$$

In the sense of α-level, $[S(t)]^\alpha = [S_l^\alpha(t), S_r^\alpha(t)] = \left[\int_0^t C_l^\alpha(s)ds, \int_0^t C_r^\alpha(s)ds\right]$, here $C_i^\alpha(t), S_i^\alpha(t)(i = l, r)$ are continuous, that is, there exist two finite constants $M_1, M_2 > 0$ such that $|C_i(t)| \le M_1, |S_i(t)| \le M_2$, $|AC_i(t)| \le M, M > 0, \forall t \in J = [0, T]$.

H2. [1] The nonlinear function $h : J \times E^n \to E^n$ is continuous and there exists a constant $d_1 > 0$, satisfying the global Lipschitz condition, such that $H_d([h(t,x)]^\alpha, [h(t,y)]^\alpha) \le d_1 H_d([x(t)]^\alpha, [y(t)]^\alpha); \forall t \in J$ and $x(t), y(t) \in E^n$.

H3. [1] If g is continuous and there exists constants $G_k, k = 1, 2, \cdots, p$, such that

$$H_d([g(x_{\tau_1}, x_{\tau_2}, \ldots, x_{\tau_p})(s)]^\alpha, [g(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_p})(s)]^\alpha)$$
$$\le \sum_{k=1}^p G_k H_d([x_{\tau_k}(s)]^\alpha, [y_{\tau_k}(s)]^\alpha), \forall s \in [-r, 0]$$

and all $x_{\tau_k}, y_{\tau_k} \in C([-r,0], E^n), k = 1, 2, \cdots, p$.

H4. [16] There exists a non-negative d_k and $d_{k'}$ such that

$$H_d([I_k(x(t_k^-))]^\alpha, [I_k(y(t_k^-))]^\alpha) \le d_k H_d([x(t)]^\alpha, [y(t)]^\alpha), k = 1, 2, \cdots, m$$

where, $\sum_{k=1}^m d_k < D$ and

$$H_d([\overline{I_k}(x(t_k^-))]^\alpha, [\overline{I_k}(y(t_k^-))]^\alpha) \le d_{k'} H_d([x(t)]^\alpha, [y(t)]^\alpha), k = 1, 2, \cdots, m$$

where, $\sum_{k=1}^m d_{k'} < \overline{D}$

for all $x, y \in E^n$ and $t \in J$.

H5. [1] The nonlinear function $f : J \times E^n \to E^n$ is continuous and there exists a constant $d_2 > 0$, satisfying the global Lipschitz condition, such that $H_d([f(t,x,x')]^\alpha, [f(t,y,y')]^\alpha) \le d_2 H_d([x(t), x'(t)]^\alpha, [y(t), y'(t)]^\alpha)$, $\forall t \in J$ and $x(t), y(t) \in E^n$.

H6. $2[M_1 \sum_{k=1}^p G_k + d_1 + T(Md_1 + M_2 d_2)] < 1$.

H7. The multi-valued map $f : J \times C([-r,0], E^n) \times C([-r,0], E^n) \to E^n$ satisfies the following conditions:

- For each $t \in J$, the function $f(t,.,.) : J \times C([-r,0], E^n) \times C([-r,0], E^n) \to E^n$ is u.s.c. and for each $x \in E^n$, the function $h(t,.) : J \times C([-r,0], E^n) \to E^n$ is measurable. Also, for each fixed $x \in \Omega$ the set $S_{(f,x,x')} = \{v \in L^1(J, E^n) : v(t) \in f(t,x,x') \text{ for a.e. } t \in J\} \ne \varphi$.

- For each positive number $l \in \mathbb{N}$, there exists a positive function $w(l)$ dependent on l such that

$$\sup_{\|x\| \le l} \|f(t,x,x')\| \le w(l)$$

and $\lim_{l\to\infty}\inf \frac{w(l)}{l} = \eta < \infty$, where $\|f(t,x,x'))\| = \sup\{\|v\| : v \in f(t,x,x')\}, \|x\|_\gamma = \sup_{0\le s\le \gamma} \|f(t,x,x'))\|$.

- f is completely continuous.

3.2 Definition

[17] If $x(t)$ is an integral solution of the problem (1.4), then $x(t)$ is given by

$$\begin{aligned} x(t) &= \varphi(t), \text{ if } t \in [-r,0] \\ x(t) &= C(t)[\varphi(0) - g(x_{\tau_1}, x_{\tau_2}, \dots, x_{\tau_p})(0)] + S(t)[y_0 - h(0,\varphi)] + h(t,x_t) \\ &+ \int_0^t AC(t-s)h(s,x_s)ds + \int_0^t S(t-s)[v(s) + Bu(s)]ds \\ &+ \sum_{0<t_k<t} C(t-t_k) I_k x(t_k^-) + \sum_{0<t_k<t} S(t-t_k)\overline{I_k} x(t_k^-) \text{ if } t \in J; \end{aligned}$$

and

$$\begin{aligned} x'(t) = {} & AS(t)[\varphi(0) - g(x_{\tau_1}, x_{\tau_2}, \dots, x_{\tau_p})(0)] + C(t)[y_0 - h(0,\varphi)] + h(t,x_t) \\ & + \int_0^t AS(t-s)h(s,x_s)ds + \int_0^t C(t-s)v(s)ds + \int_0^t C(t-s)BW^{-1}[x^1 - C(m)[\varphi(0) \\ & - g(x_{\tau_1}, x_{\tau_2}, \dots, x_{\tau_p})(0)] - S(m)[y_0 - h(0,\varphi)] - \int_0^m C(m-s)h(s,x_s)ds \\ & - \int_0^m S(m-s)v(s)ds - \sum_{0<m_k<m} C(m-m_k) I_k x(m_k^-) \\ & - \sum_{0<m_k<m} S(m-m_k)\overline{I_k} x(m_k^-)](s)ds + \sum_{0<t_k<t} AS(t-t_k) I_k x(t_k^-) \\ & + \sum_{0<t_k<t} C(t-t_k)\overline{I_k} x(t_k^-); \text{ if } t \in J \end{aligned} \tag{3.1}$$

where $v \in S_{f,x,x'} = \{v \in L^1(J,E^n) : v(t) \in f(t,x,x'(t)) \text{ for a.e. } t \in J\}$.

3.3 Definition

[7] The nonlocal problem (1.4) is said to be controllable on the interval J if there exists a control $u(t)$, such that the fuzzy solution $x(t)$ of (3.1) is controllable and satisfies

$$x[T] = x^1 \text{ i.e.,} [x(T)]^\alpha = [x^1]^\alpha \text{ where } x^1 \in E^n.$$

Before proving the controllability of system (1.4), we define the fuzzy mapping $\tilde{W}$ from $\tilde{P}(\mathcal{R})$ to E^n by

$$\tilde{W}(w)]^\alpha = \begin{cases} \int_0^T S^\alpha(T-s)w(s)ds; & w \subset \tilde{\Gamma}_u \\ 0; & otherwise \end{cases}$$

where $\tilde{P}(\mathcal{R})$ is the set of all closed compact control functions in $\mathcal{R}$ and $\tilde{\Gamma}_u$ is the closure of support u. In [2], the support Γ_u of a fuzzy number u is defined as a special case of the level set by $\Gamma_u = \{x : \mu_u(x) > 0\}$.

Then, there exists $\tilde{W}_j^{\alpha}\,(j=l,r)$,such that

$$\tilde{W}_l^{\alpha}=\int_0^T S_l^{\alpha}(T-s)Bw_l(s)ds, w_l(s)\in[u_l^{\alpha}(s),u^1(s)],$$
$$\tilde{W}_r^{\alpha}=\int_0^T S_r^{\alpha}(T-s)Bw_r(s)ds, w_r(s)\in[u^1(s),u_r^{\alpha}(s)],$$

We assume that $\tilde{W}_l^{\alpha},\tilde{W}_r^{\alpha}$ are bijective mappings. Now, the α-level set of $u(s)$ is

$$\begin{aligned}
[u(s)]^{\alpha} &= [u_l^{\alpha}(s),u_r^{\alpha}(s)]\\
&=(\tilde{W}_l^{\alpha})^{-1}((x^1)_l^{\alpha}-C_l^{\alpha}(T)[\varphi_l^{\alpha}(0)-g_l^{\alpha}(x_{\tau_1},x_{\tau_2},\ldots,x_{\tau_p})(0)]\\
&\quad -S_l^{\alpha}(T)[y_{0l}^{\alpha}-h_l^{\alpha}(0,\varphi)]-h_l^{\alpha}(T,x_{Tl}^{\alpha})-\int_0^t A_l^{\alpha}C_l^{\alpha}(T-s)h_l^{\alpha}(s,x_{sl}^{\alpha})ds\\
&\quad -\int_0^t S_l^{\alpha}(T-s)v_l^{\alpha}(s)ds-\sum\nolimits_{0<t_k<T}C_l^{\alpha}(T-t_k)I_{kl}x_l^{\alpha}(t_k^-)\\
&\quad -\sum\nolimits_{0<t_k<T}S_l^{\alpha}(T-t_k)\overline{I_{kl}}x_l^{\alpha}(t_k^-)),
\end{aligned}$$

$$\begin{aligned}
&(\tilde{W}_r^{\alpha})^{-1}((x^1)_r^{\alpha}-C_r^{\alpha}(T)[\varphi_r^{\alpha}(0)-g_r^{\alpha}(x_{\tau_1},x_{\tau_2},\ldots,x_{\tau_p})(0)]\\
&\quad -S_r^{\alpha}(T)[y_{0l}^{\alpha}-h_r^{\alpha}(0,\varphi)]-h_r^{\alpha}(T,x_{Tr}^{\alpha})-\int_0^t A_r^{\alpha}C_r^{\alpha}(T-s)h_r^{\alpha}(s,x_{sr}^{\alpha})ds\\
&\quad -\int_0^t S_r^{\alpha}(T-s)v_r^{\alpha}(s)ds-\sum\nolimits_{0<t_k<T}C_r^{\alpha}(T-t_k)I_{kr}x_r^{\alpha}(t_k^-)\\
&\quad -\sum\nolimits_{0<t_k<T}S_r^{\alpha}(T-t_k)\overline{I_{kr}}x_r^{\alpha}(t_k^-))
\end{aligned}\tag{3.2}$$

Substituting this in equation (3.1), we get an α-level set of $x(T)$ as

$$\begin{aligned}
&[x(T)]^{\alpha}=\\
&(C_l^{\alpha}(T)[\varphi_l^{\alpha}(0)-g_l^{\alpha}(x_{\tau_1},x_{\tau_2},\ldots,x_{\tau_p})(0)]+S_l^{\alpha}(T)[y_{0l}^{\alpha}-h_l^{\alpha}(0,\varphi)]+h_l^{\alpha}(T,x_{Tl}^{\alpha})\\
&+\int_0^t A_l^{\alpha}C_l^{\alpha}(T-s)h_l^{\alpha}(s,x_{sl}^{\alpha})ds+\int_0^t S_l^{\alpha}(T-s)v_l^{\alpha}(s)ds+\sum\nolimits_{0<t_k<T}C_l^{\alpha}(T-t_k)I_{kl}x_l^{\alpha}(t_k^-)\\
&+\sum\nolimits_{0<t_k<T}S_l^{\alpha}(T-t_k)\overline{I_{kl}}x_l^{\alpha}(t_k^-)+\int_0^t S_l^{\alpha}(T-s)(\tilde{W}_l^{\alpha})^{-1}((x^1)_l^{\alpha}-C_l^{\alpha}(T)[\varphi_l^{\alpha}(0)-g_l^{\alpha}(x_{\tau_1},x_{\tau_2},\cdots,x_{\tau_p})(0)]\\
&-S_l^{\alpha}(T)[y_{0l}^{\alpha}-h_l^{\alpha}(0,\varphi)]-h_l^{\alpha}(T,x_{Tl}^{\alpha})-\int_0^t A_l^{\alpha}C_l^{\alpha}(T-s)h_l^{\alpha}(s,x_{sl}^{\alpha})ds-\int_0^t S_l^{\alpha}(T-s)v_l^{\alpha}(s)ds\\
&-\sum\nolimits_{0<t_k<T}C_l^{\alpha}(T-t_k)I_{kl}x_l^{\alpha}(t_k^-)-\sum\nolimits_{0<t_k<T}S_l^{\alpha}(T-t_k)\overline{I_{kl}}x_l^{\alpha}(t_k^-))ds,
\end{aligned}$$

$$\begin{aligned}
&(C_r^{\alpha}(T)[\varphi_r^{\alpha}(0)-g_r^{\alpha}(x_{\tau_1},x_{\tau_2},\ldots,x_{\tau_p})(0)]+S_r^{\alpha}(T)[y_{0r}^{\alpha}-h_r^{\alpha}(0,\varphi)]+h_r^{\alpha}(T,x_{Tr}^{\alpha})\\
&+\int_0^t A_r^{\alpha}C_r^{\alpha}(T-s)h_r^{\alpha}(s,x_{sr}^{\alpha})ds+\int_0^t S_r^{\alpha}(T-s)v_r^{\alpha}(s)ds\\
&+\sum\nolimits_{0<t_k<T}C_r^{\alpha}v_l^{\alpha}(s)(T-t_k)I_{kr}x_r^{\alpha}(t_k^-)+\sum\nolimits_{0<t_k<T}S_r^{\alpha}(T-t_k)\overline{I_{kr}}(x_r^{\alpha}(t_k^-)\\
&+\int_0^t S_r^{\alpha}(T-s)(\tilde{W}_r^{\alpha})^{-1}((x^1)_r^{\alpha}-C_r^{\alpha}(T)[\varphi_r^{\alpha}(0)-g_r^{\alpha}(x_{\tau_1},x_{\tau_2},\cdots,x_{\tau_p}))(0)]\\
&-S_r^{\alpha}(T)[y_{0r}^{\alpha}-h_r^{\alpha}(0,\varphi)]-h_r^{\alpha}(T,x_{Tr}^{\alpha})-\int_0^t A_r^{\alpha}C_r^{\alpha}(T-s)h_r^{\alpha}(s,x_{sr}^{\alpha})ds\\
&-\int_0^t S_r^{\alpha}(T-s)v_r^{\alpha}(s)ds-\sum\nolimits_{0<t_k<T}C_r^{\alpha}(T-t_k)I_{kr}x_r^{\alpha}(t_k^-)-\sum\nolimits_{0<t_k<T}S_r^{\alpha}(T-t_k)\overline{I_{kr}}x_r^{\alpha}(t_k^-))ds\\
&=[(x^1)_l^{\alpha},(x^1)_r^{\alpha}]=[x^1]^{\alpha}.
\end{aligned}$$

Hence, the fuzzy solution *x(t)* for equation (3.1) satisfies $[x(T)]^\alpha = [x^1]^\alpha$. Now for each *x(t)* and $t \in J$, define

$$\begin{aligned}\Phi(x(t)) &= C(t)[\varphi(0) - g(x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_p})(0)] + S(t)[y_0 - h(0,\varphi)] + h(t, x_t) \\ &+ \int_0^t AC(t-s)h(s,x_s)ds + \int_0^t S(t-s)v(s)ds + \sum_{0<t_k<t} C(t-t_k)I_k x(t_k^-) \\ &+ \sum_{0<t_k<t} S(t-t_k)\overline{I_k}x(t_k^-) + \int_0^t S(t-s)(\tilde{W})^{-1}((x^1) - C(T)[\varphi(0) - g((x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_p})(0)] \\ &- S(T)[y_0 - h(0,\varphi)] - h(T, x_T) - \int_0^T A(T-s)h(s,x_s)ds - \int_0^T S(T-s)v(s)ds \\ &- \sum_{0<t_k<T} C(T-t_k)I_k x(t_k^-) - \sum_{0<t_k<T} S(T-t_k)I_k x(t_k^-)ds\end{aligned} \tag{3.3}$$

where $(\tilde{W})^{-1}$ satisfies the previous statements.

Observe $\Phi(x(t)) = [x^1]$, which represents that the control $u(t)$ steers the system (3.1) from the arbitrary stage to x^1 in time T, given that there must exist a fixed point of the nonlinear operator Φ.

Similarly,

$$\begin{aligned}\Phi(y(t)) &= C(t)[\varphi(0) - g(y_{\tau_1} y_{\tau_2}, \cdots, y_{\tau_p})(0)] + S(t)[y_0 - h(0,\varphi)] + h(t, y_t) \\ &+ \int_0^t AC(t-s)h(s,y_s)ds + \int_0^t S(t-s)v(s)ds + \sum_{0<t_k<t} C(t-t_k)I_k y(t_k^-) \\ &+ \sum_{0<t_k<t} S(t-t_k)\overline{I_k}y(t_k^-) + \int_0^t S(t-s)(\tilde{H})^{-1}((x^1) - C(T)[\varphi(0) - g(y_{\tau_1}, y_{\tau_2}, \cdots, y_{\tau_p})(0)] \\ &- S(T)[y_0 - h(0,\varphi)] - h(T, y_T) - \int_0^T A(T-s)h(s,y_s)ds - \int_0^T S(T-s)v(s)ds \\ &- \sum_{0<t_k<T} C(T-t_k)I_k x(t_k^-) - \sum_{0<t_k<T} S(T-t_k)I_k y(t_k^-)ds\end{aligned} \tag{3.4}$$

The controllability of fuzzy solutions for the neutral impulsive functional differential equation with nonlocal conditions is discussed in the following theorem.

Theorem: [7] Equation (3.4) is controllable if the hypothesis (H1–H6) is satisfied.

Proof: For $x, y \in \Omega'$

$$\begin{aligned}H_d([\Phi(x(t))]^\alpha, [\Phi(y(t))]^\alpha) &= H_d[[C(t)[\phi(0) - g(x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_p})(0)] + S(t)[y_0 - h(0,\phi)] + h(t, x_t) \\ &+ \int_0^t AC(t-s)h(s,x_s)ds + \int_0^t S(t-s)v(s)ds + \sum_{0<t_k<t} C(t-t_k)I_k x(t_k^-) + \sum_{0<t_k<t} S(t-t_k)\overline{I_k}x(t_k^-) \\ &+ \int_0^t S(t-s)(\tilde{W})^{-1}((x^1) - C(T)[\phi(0) - g((x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_p})(0)] - S(T)[y_0 - h(0,\phi)] - h(T, x_T) \\ &- \int_0^T A(T-s)h(s,x_s)ds - \int_0^T S(T-s)v(s)ds - \sum_{0<t_k<T} C(T-t_k)I_k x(t_k^-) - \sum_{0<t_k<T} S(T-t_k)I_k x(t_k^-)ds]^\alpha, \\ &[C(t)[\phi(0) - g(y_{\tau_1} y_{\tau_2}, \cdots, y_{\tau_p})(0)] + S(t)[y_0 - h(0,\phi)] + h(t, y_t) + \int_0^t AC(t-s)h(s,y_s)ds + \int_0^t S(t-s)v(s)ds \\ &+ \sum_{0<t_k<t} C(t-t_k)I_k y(t_k^-) + \sum_{0<t_k<t} S(t-t_k)\overline{I_k}y(t_k^-) + \int_0^t S(t-s)(\tilde{W})^{-1}((x^1) - C(T)[\phi(0) - g(y_{\tau_1}, y_{\tau_2}, \cdots, y_{\tau_p})(0)] \\ &- S(T)[y_0 - h(0,\phi)] - h(T, y_T) - \int_0^T AC(T-s)h(s,y_s)ds - \int_0^T S(T-s)v(s)ds - \sum_{0<t_k<T} C(T-t_k)I_k x(t_k^-) \\ &- \sum_{0<t_k<T} S(T-t_k)I_k y(t_k^-)ds]^\alpha]\end{aligned}$$

$$= H_d([C(t)[\phi(0) - g(x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_p})(0)]]^\alpha + [S(t)[y_0 - h(0,\phi)]^\alpha + H_d[h(t,x_t)]^\alpha + \left[\int_0^t AC(t-s)h(s,x_s)ds\right]^\alpha$$

$$+ \left[\int_0^t S(t-s)v(s)ds\right]^\alpha + \left[\sum_{0<t_k<t} C(t-t_k)I_k x(t_k^-)\right]^\alpha + \left[\sum_{0<t_k<T} S(T-t_k)\overline{I_k}x(t_k^-)ds\right]^\alpha$$

$$+ \left[\int_0^t S(t-s)(\tilde{W})^{-1}((x^1) - C(T)[\phi(0) - g(x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_p})(0)\right] - S(T)[y_0 - h(0,\phi)] - h(T, x_T)$$

$$- \int_0^T A(T-s)h(s,x_s)ds - \int_0^T S(T-s)v(s)ds - \sum_{0<t_k<T} C(T-t_k)I_k x(t_k^-) - \sum_{0<t_k<T} S(T-t_k)I_k x(t_k^-)ds]^\alpha,$$

$$[C(t)[\varphi(0) - g(y_{\tau_1} y_{\tau_2}, \cdots, y_{\tau_p})(0)]^\alpha + [S(t)[y_0 - h(0,\varphi)]^\alpha + H_d[h(t,y_t)]^\alpha + \left[\int_0^t AC(t-s)h(s,y_s)ds\right]^\alpha$$

$$+ \left[\sum\nolimits_{0<t_k<t} C(t-t_k)I_k y(t_k^-)\right]^\alpha + \left[\sum\nolimits_{0<t_k<T} S(T-t_k)\overline{I_k}y(t_k^-)ds\right]^\alpha$$

$$+ \left[\int_0^T S(T-s)v(s)ds\right]^\alpha + \left[\int_0^t S(t-s)(\tilde{W})^{-1}((x^1) - C(T)[\varphi(0) - g(y_{\tau_1}, y_{\tau_2}, \cdots, y_{\tau_p})(0)]\right.$$

$$- S(T)[y_0 - h(0,\varphi)] - h(T, y_T)\int_0^T A(T-s)h(s,y_s)ds - \int_0^T S(T-s)v(s)ds$$

$$\left.\left. - \sum\nolimits_{0<t_k<T} C(T-t_k)I_k x(t_k^-) - \sum\nolimits_{0<t_k<T} S(T-t_k)I_k y(t_k^-)ds\right]^\alpha\right)$$

$$= H_d([C(t)g(x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_p})(0)]^\alpha, [C(t)g(y_{\tau_1}, y_{\tau_2}, \cdots, y_{\tau_p})(0)]^\alpha)$$

$$+ H_d([h(t,x_t)]^\alpha, [h(t,y_t)]^\alpha) + H_d\left(\left[\int_0^t AC(t-s)h(s,x_s)ds\right]^\alpha, \left[\int_0^t AC(t-s)h(s,y_s)ds\right]^\alpha\right)$$

$$+ H_d\left(\left[\int_0^t S(t-s)v(s)ds\right]^\alpha, \left[\int_0^t S(t-s)v(s)ds\right]^\alpha\right)$$

$$+ H_d\left(\left[\sum\nolimits_{0<t_k<t} C(t-t_k)I_k x(t_k^-)\right]^\alpha, \left[\sum\nolimits_{0<t_k<t} C(t-t_k)I_k y(t_k^-)\right]^\alpha\right)$$

$$+ H_d\left(\left[\sum\nolimits_{0<t_k<t} S(t-t_k)I_k y(t_k^-)\right]^\alpha, \left[\sum\nolimits_{0<t_k<t} S(t-t_k)\overline{I_k}y(t_k^-)\right]^\alpha\right)$$

$$+ H_d\left(\int_0^t S_l^\alpha(t-s)B\tilde{W}_l^\alpha)^{-1}[(x^1) - C(T)[\varphi(0) - g(x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_p})(0)]\right.$$

$$- S(T)[y_0 - h(0,\varphi)] - h(T,x_T) - \int_0^T A(T-s)h(s,x_s)ds - \int_0^T S(T-s)v(s)ds$$

$$- \sum\nolimits_{0<t_k<T} C(T-t_k)I_k x(t_k^-) - \sum\nolimits_{0<t_k<T} S(T-t_k)\overline{I_k}x(t_k^-)\Big]_l^\alpha (s)ds, \int_0^t S_r^\alpha(t-s)B(\tilde{W}_r^\alpha)^{-1}[(x^1)$$

$$- C(T)[\varphi(0) - g(x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_p})(0)] - S(T)[y_0 - h(0,\varphi)] - h(T,x_T)$$

$$- \int_0^T A(T-s)h(s,x_s)ds - \int_0^T S(T-s)v(s)ds - \sum\nolimits_{0<t_k<T} C(T-t_k)I_k x(t_k^-)$$

$$- \sum\nolimits_{0<t_k<T} S(T-t_k)\overline{I_k}x(t_k^-)]_r^\alpha(s)ds], \left[\int_0^t S_l^\alpha(t-s)(\tilde{W}_l^\alpha - 1)[(x^1) - C(T)[\varphi(0)\right.$$

$$- g(y_{\tau_1} y_{\tau_2}, \cdots, y_{\tau_p})(0)] - S(T)[y_0 - h(0,\varphi)] - h(T, y_T) - \int_0^T A(T-s)h(s,y_s)ds$$

$$-\int_0^T S(T-s)v(s)ds-\sum_{0<t_k<T}C(T-t_k)I_k y(t_k^-)-\sum_{0<t_k<T}S(T-t_k)\overline{I_k}y(t_k^-)]_l^\alpha(s)ds,$$

$$\int_0^t S_r^\alpha(t-s)(\tilde{W}_r^\alpha)^{-1}[(x^1)-C(T)[\varphi(0)-g(y_{\tau_1}y_{\tau_2},\cdots,y_{\tau_p})(0)]-S(T)[y_0-h(0,\varphi)]-h(T,y_T)$$

$$-\int_0^T A(T-s)h(s,y_s)ds-\int_0^T S(T-s)v(s)ds-\sum_{0<t_k<T}C(T-t_k)I_k y(t_k^-)-\sum_{0<t_k<T}S(T-t_k)\overline{I_k}y(t_k^-)]_r^\alpha(s)ds]\Big)$$

$$\le H_d([(C(t))_l^\alpha g_l^\alpha(x_{\tau_1},x_{\tau_2},\cdots,x_{\tau_p})(0)],[(C(t))_r^\alpha g_r^\alpha(x_{\tau_1},x_{\tau_2},\cdots,x_{\tau_p})(0)],$$

$$[(C(t))_l^\alpha g_l^\alpha(y_{\tau_1}y_{\tau_2},\cdots,y_{\tau_p})(0)],[(C(t))_r^\alpha g_r^\alpha(y_{\tau_1}y_{\tau_2},\cdots,y_{\tau_p})(0)])$$

$$+H_d([h_l^\alpha(t,x_{tl}^\alpha),h_r^\alpha(t,x_{tr}^\alpha)],[h_l^\alpha(t,y_{tl}^\alpha),h_r^\alpha(t,y_{tr}^\alpha)])$$

$$+H_d\left[\int_0^t A_l^\alpha C_l^\alpha(t-s)h_l^\alpha(s,x_{sl}^\alpha)ds,\int_0^t A_r^\alpha C_r^\alpha(t-s)h_r^\alpha(s,x_{sr}^\alpha)ds\right],$$

$$\left[\int_0^t A_l^\alpha C_l^\alpha(t-s)h(s,y_{sl}^\alpha)ds,\int_0^t A_r^\alpha C_r^\alpha(t-s)h(s,y_{sr}^\alpha)ds\right]$$

$$+H_d\left[\int_0^t S_l^\alpha(t-s)v_l^\alpha(s)ds\right],\left[\int_0^t S_r^\alpha(t-s)v_r^\alpha(s)ds\right],\left[\int_0^t S_l^\alpha(t-s)v_l^\alpha(s)ds\right],\left[\int_0^t S_r^\alpha(t-s)v_r^\alpha(s)ds\right]$$

$$+H_d\left[\sum_{0<t_k<t}C_l^\alpha(t-t_k)I_{kl}^\alpha x(t_k^-),\sum_{0<t_k<t}C_r^\alpha t-t_k I_{kr}^\alpha x(t_k^-)\right],$$

$$\left[\sum_{0<t_k<t}C_l^\alpha(t-t_k)I_{kl}^\alpha y(t_k^-),\sum_{0<t_k<t}C_r^\alpha(t-t_k)I_{kr}^\alpha y(t_k^-)\right]$$

$$+H_d\left[\sum_{0<t_k<t}S_l^\alpha(t-t_k)\overline{I_{kl}}^\alpha x(t_k^-),\sum_{0<t_k<t}S_r^\alpha(t-t_k)\overline{I_{kr}}^\alpha x(t_k^-)\right],$$

$$\left[\sum_{0<t_k<t}S_l^\alpha(t-t_k)\overline{I_{kl}}^\alpha y(t_k^-),\sum_{0<t_k<t}S_r^\alpha(t-t_k)\overline{I_{kr}}^\alpha y(t_k^-)\right]$$

$$H_d([(\tilde{W}_l^\alpha)(\tilde{W}_l^\alpha)^{-1}[(x^1)-C(T)[\varphi(0)-g(x_{\tau_1},x_{\tau_2},\cdots,x_{\tau_p})(0)]-S(T)[y_0-h(0,\varphi)]-h(T,x_T)$$

$$-\int_0^T A(T-s)h(s,x_s)ds-\int_0^T S(T-s)v(s)ds-\sum_{0<t_k<T}C(T-t_k)I_k x(t_k^-)-\sum_{0<t_k<T}S(T-t_k)\overline{I_k}x(t_k^-)\Big]_l^\alpha\Big],$$

$$[(\tilde{W}_r^\alpha)(\tilde{W}_r^\alpha)^{-1}[(x^1)-C(T)[\varphi(0)-g(x_{\tau_1},x_{\tau_2},\cdots,x_{\tau_p})(0)]-S(T)[y_0-h(0,\varphi)]-h(T,x_T)$$

$$-\int_0^T A(T-s)h(s,x_s)ds-\int_0^T S(T-s)v(s)ds-\sum_{0<t_k<T}C(T-t_k)I_k x(t_k^-)-\sum_{0<t_k<T}S(T-t_k)\overline{I_k}x(t_k^-)\Big]_r^\alpha\Big],$$

$$[(\tilde{W}_l^\alpha)(\tilde{W}_l^\alpha)^{-1}[(x^1)-C(T)[\varphi(0)-g(y_{\tau_1},y_{\tau_2},\cdots,y_{\tau_p})(0)]-S(T)[y_0-h(0,\varphi)]-h(T,y_T)$$

$$-\int_0^T A(T-s)h(s,y_s)ds-\int_0^T S(T-s)v(s)ds-\sum_{0<t_k<T}C(T-t_k)I_k y(t_k^-)-\sum_{0<t_k<T}S(T-t_k)\bar{I}_k y(t_k^-)\Big]_l^\alpha\Big],$$

$$[(\tilde{W}_r^\alpha)(\tilde{W}_r^\alpha)^{-1}[[(x^1)-C(T)[\varphi(0)-g(y_{\tau_1},y_{\tau_2},\cdots,y_{\tau_p})(0)]-S(T)[y_0-h(0,\varphi)]-h(T,y_T)$$

$$-\int_0^T A(T-s)h(s,y_s)ds-\int_0^T S(T-s)v(s)ds-\sum_{0<t_k<T}C(T-t_k)I_k y(t_k^-)-\sum_{0<t_k<T}S(T-t_k)\bar{I}_k y(t_k^-)\Big]_r^\alpha$$

$$\le 2M_1\sum_{k=1}^p G_k H_d([x_{\tau_k}(0)]^\alpha,[y_{\tau_k}(0)]^\alpha)+2(d_1)H_d([x(t)]^\alpha,[y(t)]^\alpha)+Md_1\int_0^t H_d([x(t)]^\alpha,[y(t)]^\alpha)ds$$

$$+M_2d_2\int_0^t H_d([x(t)]^\alpha,[y(t)]^\alpha)ds+Md_1\int_0^T H_d([x(t)]^\alpha,[y(t)]^\alpha)ds+M_2d_2\int_0^T H_d([x(t)]^\alpha,[y(t)]^\alpha)ds$$

Therefore,

$$d_\infty(\Phi x(t),\Phi y(t)) = H_d([\Phi x(t)]^\alpha,[\Phi y(t)]^\alpha) \le \sum_{k=1}^{p} G_k d_\infty(x_{\tau_k}, y_{\tau_k}) + 2(d_1)d_\infty(x(t),y(t))$$
$$+Md_1\int_0^t d_\infty(x(s),y(s))ds + M_2d_2\int_0^t d_\infty(x(s),y(s))ds + Md_1\int_0^T d_\infty(x(s),y(s))ds + M_2d_2\int_0^t d_\infty(x(s),y(s))ds$$

Hence, $H_1(\Phi(x),\Phi(y)) = \sup_{0\le t\le T} d_\infty(\Phi(x(t)),\Phi(y(t)))$

$$\le 2M_1\sum_{k=1}^{p} G_k \sup_{0\le t\le T} d_\infty(x_{\tau_k}, y_{\tau_k}) + 2(d_1)\sup_{0\le t\le T} d_\infty(x(t),y(t))$$
$$+ Md_1\sup_{0\le t\le T}\int_0^t d_\infty(x(s),y(s))ds + M_2d_2\sup_{0\le t\le T}\int_0^t d_\infty(x(s),y(s))ds$$
$$=\left[2M_1\sum_{k=1}^{p} G_k + 2(d_1) + 2T(Md_1 + M_2d_2)\right]H_1(x,y).$$

By condition (H6), Φ is a contraction mapping. Using the Banach fixed point theorem, equation (3.3) has a unique fixed point $x \in \Omega'$. Hence the System (1.4) is controllable on J.

Similarly, we proceed for x',

$$\begin{aligned}[u(s)]^\alpha &= [u_l^\alpha(s), u_r^\alpha(s)]\\ &= ((\tilde{W}_l^\alpha)^{-1}[x^1 - C_l^\alpha(T)[\varphi_l^\alpha(0) - g_l^\alpha(x_{\tau_1}, x_{\tau_2},\ldots,x_{\tau_p})(0)] - S_l^\alpha(T)[y_{0l}^\alpha - h_l^\alpha(0,\varphi)]\\ &-\int_0^t C_l^\alpha(T-s)h_l^\alpha(s,x_{sl}^\alpha)ds - \int_0^t S_l^\alpha(T-s)v_l^\alpha(s)ds - \sum\nolimits_{0<t_k<T} C_l^\alpha(T-t_k)I_{kl}x_l^\alpha(t_k^-)\\ &-\sum\nolimits_{0<t_k<T} S_l^\alpha(T-t_k)\overline{I_{kl}}x_l^\alpha(t_k^-)](s)ds, ((\tilde{W}_r^\alpha)^{-1}[x^1 - C_r^\alpha(T)[\varphi_r^\alpha(0) - g_r^\alpha(x_{\tau_1}, x_{\tau_2},\ldots,x_{\tau_p})(0)]\\ &-S_r^\alpha(T)[y_{0r}^\alpha - h_r^\alpha(0,\varphi)] - \int_0^t C_r^\alpha(T-s)h_r^\alpha(s,x_{sr}^\alpha)ds - \int_0^t S_r^\alpha(T-s)v_r^\alpha(s)ds\\ &-\sum\nolimits_{0<t_k<T} C_r^\alpha(T-t_k)I_{kr}x_r^\alpha(t_k^-) - \sum\nolimits_{0<t_k<T} S_r^\alpha(T-t_k)\overline{I_{kr}}x_r^\alpha(t_k^-)](s)ds\end{aligned} \tag{3.5}$$

Substituting this in equation (3.1), we get an α-level set of $x'(T)$ as

$$\begin{aligned}[x'(T)] &= A_l^\alpha S_l^\alpha(T)[\varphi_l^\alpha(0) - g_l^\alpha(x_{\tau_1}, x_{\tau_2},\cdots,x_{\tau_p})(0)] + C_l^\alpha(T)[y_{0l}^\alpha - h_l^\alpha(0,\varphi)] + h_l^\alpha(T, x_{Tl}^\alpha)\\ &+\int_0^t A_l^\alpha S_l^\alpha(T-s)h_l^\alpha(s,x_{sl}^\alpha)ds + \int_0^t C_l^\alpha(T-s)v(s)ds + \sum\nolimits_{0<t_k<T} A_l^\alpha S_l^\alpha(T-t_k)I_{kl}x(t_k^-)\\ &+\sum\nolimits_{0<t_k<T} C_l^\alpha(T-t_k)\overline{I_k}x(t_k^-) + \int_0^t C_l^\alpha(T-s)B(\tilde{W}_l^\alpha)^{-1}[x^1 - C_l^\alpha(T)[\varphi_l^\alpha(0) - g_l^\alpha(x_{\tau_1}, x_{\tau_2},\cdots,x_{\tau_p})(0)]\\ &-S_l^\alpha(T)[y_{0l}^\alpha - h_l^\alpha(0,\varphi)] - \int_0^t C_l^\alpha(T-s)h_l^\alpha(s,x_{sl}^\alpha)ds - \int_0^t S_l^\alpha(T-s)v_l^\alpha(s)ds\\ &-\sum\nolimits_{0<t_k<T} C_l^\alpha(T-t_k)I_{kl}x_l^\alpha(t_k^-) - \sum\nolimits_{0<t_k<T} S_l^\alpha(T-t_k)\overline{I_{kl}}x_l^\alpha(t_k^-)](s)ds,\end{aligned}$$

$$\begin{aligned}&A_r^\alpha S_r^\alpha(T)[\varphi_r^\alpha(0) - g_r^\alpha(x_{\tau_1}, x_{\tau_2},\cdots,x_{\tau_p})(0)] + C_r^\alpha(T)[y_{0r}^\alpha - h_r^\alpha(0,\varphi)] + h_r^\alpha(T, x_{Tr}^\alpha)\\ &+\int_0^t A_r^\alpha S_r^\alpha(T-s)h_r^\alpha(s,x_{sr}^\alpha)ds + \int_0^t C_r^\alpha(T-s)v(s)ds + \sum\nolimits_{0<t_k<T} A_r^\alpha S_r^\alpha(T-t_k)I_{kr}x_r^\alpha(t_k^-)\\ &+\sum\nolimits_{0<t_k<T} C_r^\alpha(T-t_k)\overline{I_{kr}}(x_r^\alpha(t_k^-) + \int_0^t C_r^\alpha(T-s)(\tilde{W}_r^\alpha)^{-1}((x^1)_l^\alpha - C_r^\alpha(T)[\varphi_l^\alpha(0) - g_r^\alpha(x_{\tau_1}, x_{\tau_2},\cdots,x_{\tau_p})(0)]\\ &-S_l^\alpha(T)[y_{0l}^\alpha - h_r^\alpha(0,\varphi)] - \int_0^t C_r^\alpha(T-s)h_r^\alpha(s,x_{sr}^\alpha)ds - \int_0^t S_r^\alpha(T-s)v_r^\alpha(s)ds\\ &-\sum\nolimits_{0<t_k<T} C_r^\alpha(T-t_k)I_{kr}x_r^\alpha(t_k^-) - \sum\nolimits_{0<t_k<T} S_r^\alpha(T-t_k)\overline{I_{kr}}x_l^\alpha(t_k^-)](ds) = [(x^1)_l^\alpha, (x^1)_r^\alpha] = [x^1]^\alpha.\end{aligned}$$

Hence, the fuzzy solution *x(t)* for equation (3.1) satisfies $[x'(T)]^\alpha = [x^1]^\alpha$. Now for each $x'(t)$ and $t \in J$. define

$$
\begin{aligned}
\Phi(x'(t)) &= AS(t)[\varphi(0) - g(x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_p})(0)] + C(t)[y_0 - h(0,\varphi)] + h(t, x_t) \\
&+ \int_0^t AS(t-s)h(s,x_s)ds + \int_0^t C(t-s)v(s)ds + \sum_{0<t_k<t} AS(t-t_k)I_k x(t_k^-) \\
&+ \sum_{0<t_k<t} C(t-t_k)\overline{I_k} x(t_k^-) + \int_0^t C(t-s)B(\tilde{W})^{-1}((x^1) - C(m)[\varphi(0) \\
&- g(x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_p})(0)] - S(m)[y_0 - h(0,\varphi)] - \int_0^m C(m-s)h(s,x_s)ds \\
&- \int_0^m S(m-s)v(s)ds - \sum_{0<m_k<m} C(m-m_k)I_k x(m_k^-) \\
&- \sum_{0<m_k<m} S(m-m_k)I_k x(m_k^-))ds
\end{aligned}
\tag{3.6}
$$

where $(\tilde{W})^{-1}$ satisfies the Equation (3.5).

Observe $\Phi(x(t)) = [x^1]$, which represents that the control $u(t)$ steers the system (3.1) from the arbitrary stage to x^1 in time T, given that there must exist a fixed point of the nonlinear operator Φ.

Similarly,

$$
\begin{aligned}
\Phi(y'(t)) &= AS(t)[\varphi(0) - g(y_{\tau_1} y_{\tau_2}, \cdots, y_{\tau_p})(0)] + C(t)[y_0 - h(0,\varphi)] + h(t, y_t) \\
&+ \int_0^t AS(t-s)h(s,y_s)ds + \int_0^t C(t-s)v(s)ds + \sum_{0<t_k<t} AS(t-t_k)I_k y(t_k^-) \\
&+ \sum_{0<t_k<t} C(t-t_k)\overline{I_k} y(t_k^-) + \int_0^t C(t-s)B(\tilde{W})^{-1}[(x^1) - C(m)[\varphi(0) - g(y_{\tau_1}, y_{\tau_2}, \cdots, y_{\tau_p})(0)] \\
&- S(m)[y_0 - h(0,\varphi)] - \int_0^m C(m-s)h(s,y_s)ds - \int_0^m S(m-s)v(s)ds \\
&- \sum_{0<m_k<m} C(m-m_k)I_k x(m_k^-) - \sum_{0<m_k<m} S(m-m_k)I_k y(m_k^-)]ds
\end{aligned}
$$

For $x', y' \in \Omega'$,

$$
\begin{aligned}
&H_d([\Phi(x'(t))]^\alpha, [\Phi(y'(t))]^\alpha) = [[AS(t)[\varphi(0) - g(x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_p})(0)] + C(t)[y_0 - h(0,\varphi)] + h(t, x_t) \\
&+ \int_0^t AS(t-s)h(s,x_s)ds + \int_0^t C(t-s)v(s)ds + \sum_{0<t_k<t} AS(t-t_k)I_k x(t_k^-) \\
&+ \sum_{0<t_k<t} C(t-t_k)\overline{I_k} x(t_k^-) + \int_0^t C(t-s)B(\tilde{W})^{-1}((x^1) - C(m)[\varphi(0) - g(x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_p})(0)] \\
&- S(m)[y_0 - h(0,\varphi)] - \int_0^m C(m-s)h(s,x_s)ds - \int_0^m S(m-s)v(s)ds \\
&- \sum_{0<m_k<m} C(m-m_k)I_k x(m_k^-) - \sum_{0<m_k<m} S(m-m_k)I_k x(m_k^-))ds]^\alpha, [AS(t)[\varphi(0) - g(y_{\tau_1}, y_{\tau_2}, \cdots, y_{\tau_p})(0)] \\
&+ C(t)[y_0 - h(0,\varphi)] + h(t, y_t) + \int_0^t AS(t-s)h(s,y_s)ds + \int_0^t C(t-s)v(s)ds \\
&+ \sum_{0<t_k<t} AS(t-t_k)I_k y(t_k^-) + \sum_{0<t_k<t} C(t-t_k)\overline{I_k} y(t_k^-) \\
&+ \int_0^t C(t-s)B(\tilde{W})^{-1}[(x^1) - C(m)[\varphi(0) - g(y_{\tau_1}, y_{\tau_2}, \cdots, y_{\tau_p}))(0)] - S(m)[y_0 - h(0,\varphi)] \\
&- \int_0^m C(m-s)h(s,y_s)ds - \int_0^m S(m-s)v(s)ds - \sum_{0<m_k<m} C(m-m_k)I_k x(m_k^-) \\
&- \sum_{0<m_k<m} S(m-m_k)I_k y(m_k^-)]ds]^\alpha].
\end{aligned}
$$

Proceeding in the similar fashion as in $x(t)$, we get,

$$d_\infty(\Phi x'(t),\Phi y'(t)) = \sup_{0\le\alpha\le1} H_d([\Phi x'(t)]^\alpha,[\Phi y'(t)]^\alpha)$$
$$\le [(M_1+M_2)\sum_{k=1}^{p} G_k + (1+M_1)d_1 + T(3M_2d_1 + 2M_1d_2 + M_2d_2)]H_1(x,y).$$

4. Example

In this section, we apply the results proved in the previous section to study the controllability of the following partial differential equation:

$$\left.\begin{aligned} x_{tt}(t,z) &= [\tilde{2}x(t,z)]_{zz} + u(t,z) + \tilde{2}tx(t,z)^2 \\ \Delta x(t_k) &= I_k x(t_k^-), t\ne t_k, k=1,2,\cdots,m, \\ \Delta x'(t_k) &= \overline{I_k} x(t_k^-), t\ne t_k, \\ x(0)+\sum_{k=1}^{p} c_k x(t_k) &= 0\in E^n, x'(0)=0 \end{aligned}\right\} \tag{4.1}$$

The α-level set of number $\tilde{0}$ is given by, $[\tilde{0}]^\alpha = [\alpha-1, 1-\alpha]$; for all $\alpha\in[0,1]$.

And the α-level set of number $\tilde{2}$ is given by, $[\tilde{2}]^\alpha = [1+\alpha, 3-\alpha]$; for all $\alpha\in[0,1]$.

Now, the α-level set of $[f(t,x_t)]^\alpha = [\tilde{2}tx(t,z)^2]^\alpha = t[1+\alpha,3-\alpha][(x_l^\alpha(t,z))^2,(x_r^\alpha(t,z))^2]$

$$=[(1+\alpha)t(x_l^\alpha(t,z))^2,(3-\alpha)t(x_r^\alpha(t,z))^2],$$

where $[x(t,z)]^\alpha = [x_l^\alpha(t,z), x_r^\alpha(t,z)]$.

Further, $H_d([f(t,x_t)]^\alpha,[f(t,y_t)]^\alpha)$

$$= H_d([(1+\alpha)t(x_l^\alpha(t,z))^2,(3-\alpha)t(x_r^\alpha(t,z))^2],[(1+\alpha)t(y_l^\alpha(t,z))^2,(3-\alpha)t(y_r^\alpha(t,z))^2])$$
$$= max((1+\alpha)t\,|(x_l^\alpha(t,z))^2-(y_l^\alpha(t,z))^2|,(3-\alpha)t\,|(x_r^\alpha(t,z))^2-(y_r^\alpha(t,z))^2|)$$
$$\le (3-\alpha)Tmax(|x_l^\alpha(t,z)-y_l^\alpha(t,z)|\,|x_l^\alpha(t,z)+y_l^\alpha(t,z)|,|x_r^\alpha(t,z)-y_r^\alpha(t,z)|\,|x_r^\alpha(t,z)+y_r^\alpha(t,z)|)$$
$$\le (3T\,|x_r^\alpha(t,z)+y_r^\alpha(t,z)|)H_d([x(t,z)]^\alpha,[x(t,z)]^\alpha)$$
$$= K_1 H_d([x(t,z)]^\alpha,[y(t,z)]^\alpha),$$

where $K_1 = (3T\,|x_r^\alpha(t,z)+y_r^\alpha(t,z)|)$ satisfies the inequality which is given in condition (H5).

Let the target state be $\tilde{2}$. Now, from the definition of fuzzy solution

$$x_l^\alpha(t) = C_l^\alpha(t)\left[(\alpha-1)-\sum_{k=1}^{p} c_k x(t_k)\right] + S_l^\alpha(t)[\alpha-1] + S_l^\alpha(t-s)[(1+\alpha)s(x_l^\alpha(t,z))^2]ds$$
$$+\int_0^t S_l^\alpha(t-s)u_l^\alpha(s)ds + \sum_{0<t_k<t} C_l^\alpha(t-t_k)I_{kl}x_l^\alpha(t_k^-) + \sum_{0<t_k<t} S_l^\alpha(t-t_k)\overline{I_{kl}}x_l^\alpha(t_k^-),$$
$$x_r^\alpha(t) = C_r^\alpha(t)\left[(1-\alpha)-\sum_{k=1}^{p} c_k x(t_k)\right] + S_r^\alpha[1-\alpha] + S_r^\alpha(t-s)[(3-\alpha)t(x_r^\alpha(t,z))^2]ds$$
$$+\int_0^t S_r^\alpha(t-s)u_r^\alpha(s)ds + \sum_{0<t_k<t} C_r^\alpha(t-t_k)I_{kr}x_r^\alpha(t_k^-) + \sum_{0<t_k<t} S_r^\alpha(t-t_k)\overline{I_{kr}}x_r^\alpha(t_k^-),$$

Now the α-level set of $u(s)$ is $[u(s)]^\alpha = [u_l^\alpha(s), u_r^\alpha(s)]$
where

$$u_l^\alpha(s) = (\tilde{W}_l^\alpha)^{-1}((1+\alpha) - C_l^\alpha(T)[(\alpha-1) - \sum_{k=1}^{p} c_k x(t_k)] - S_l^\alpha(T)[\alpha-1]$$
$$-\int_0^t S_l^\alpha(T-s)[(1+\alpha)s(x_l^\alpha(t,z))^2]ds - \sum_{0<t_k<T} C_l^\alpha(T-t_k) I_{kl} x_l^\alpha(t_k^-)$$
$$-\sum_{0<t_k<T} S_l^\alpha(T-t_k)\overline{I_{kl}} x_l^\alpha(t_k^-))(s),$$

and

$$u_r^\alpha(s) = (\tilde{W}_r^\alpha)^{-1}(3-\alpha) - C_r^\alpha(T)[(\alpha-1) - \sum_{k=1}^{p} c_k x(t_k)] - S_r^\alpha(T)[1-\alpha]$$
$$-\int_0^t S_r^\alpha(T-s)[(3-\alpha)t(x_r^\alpha(t,z))^2]ds - \sum_{0<t_k<T} C_r^\alpha(T-t_k) I_{kr} x_r^\alpha(t_k^-)$$
$$-\sum_{0<t_k<T} S_r^\alpha(T-t_k)\overline{I_{kr}} x_r^\alpha(t_k^-))(s)$$

Then the α-level set of $x(T)$ is given by,

$$[x(T)]^\alpha = [x_l^\alpha(T), x_r^\alpha(T)] = (C_l^\alpha(T)[(\alpha-1) - \sum_{0<t_k<T} c_k x(t_k)] + S_l^\alpha(T)[\alpha-1]$$
$$+\int_0^T S_l^\alpha(T-s)[(1+\alpha)t(x_l^\alpha(t,z))^2]ds + \sum_{0<t_k<T} C_l^\alpha(T-t_k) I_{kl} x_l^\alpha(t_k^-)$$
$$+\sum_{0<t_k<T} S_l^\alpha(T-t_k)\overline{I_{kl}} x_l^\alpha(t_k^-) + (\tilde{W}_l^\alpha)(\tilde{W}_l^\alpha)^{-1}((1+\alpha) - C_l^\alpha(T)[(\alpha-1)$$
$$-\sum_{k=1}^{p} c_k x(t_k)] - S_l^\alpha(T)[\alpha-1] - \int_0^t S_l^\alpha(T-s)[(1+\alpha)s(x_l^\alpha(t,z))^2]ds$$
$$-\sum_{0<t_k<T} C_l^\alpha(T-t_k) I_{kl} x_l^\alpha(t_k^-) - \sum_{0<t_k<T} S_l^\alpha(T-t_k)\overline{I_{kl}} x_l^\alpha(t_k^-)),$$
$$(C_r^\alpha(T)[(1-\alpha) - \sum_{k=1}^{p} c_k x(t_k)] + S_r^\alpha(T)[1-\alpha] + S_r^\alpha(T-s)[(3-\alpha)t(x_r^\alpha(t,z))^2]ds$$
$$+\int_0^t S_r^\alpha(T-s)u_r^\alpha(s)ds + \sum_{0<t_k<T} C_r^\alpha(T-t_k) I_{kr} x_r^\alpha(t_k^-)$$
$$+\sum_{0<t_k<T} S_r^\alpha(T-t_k)\overline{I_{kr}} x_r^\alpha(t_k^-) + ((\tilde{W}_r^\alpha)((\tilde{W}_r^\alpha)^{-1}((3-\alpha) - C_r^\alpha(T)[(1-\alpha)$$
$$-\sum_{k=1}^{p} c_k x(t_k)] - S_r^\alpha(T)[1-\alpha] - \int_0^t S_r^\alpha(T-s)[(3-\alpha)t(x_r^\alpha(t,z))^2]ds$$
$$-\sum_{0<t_k<T} C_r^\alpha(T-t_k) I_{kr} x_r^\alpha(t_k^-) - \sum_{0<t_k<T} S_r^\alpha(T-t_k)\overline{I_{kr}} x_r^\alpha(t_k^-)),$$
$$= [1+\alpha, 3-\alpha] = [2]^\alpha = [x^1]^\alpha$$

Hence, the fuzzy solution $x(t)$ for equation (3.1) satisfies $[x(T)]^\alpha = [x^1]^\alpha$. Thus all the conditions stated in Theorem 3.1 are satisfied. So the system (4.1) is controllable on J.

5. Application

Consider a coil spring suspended from the ceiling with 8-lb weight placed upon the lower end of the spring. Stretching the spring $\frac{1}{2}$ ft from its equilibrium position and then pulled down the weight about $\frac{\tilde{1}}{4}$ft $[i.e.(\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{2}; 1)ft]$ below its equilibrium position and released at $t = 0$ with initial velocity

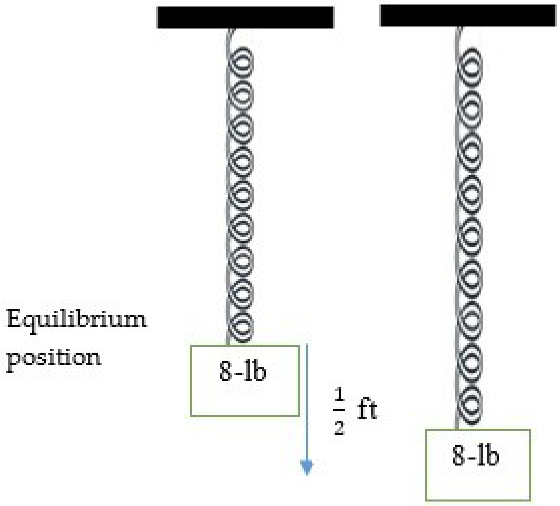

Figure 1: Represents the coil spring-suspended with 8-lb mass

1 ft/sec [$i.e.(\frac{1}{2},1,2,\frac{5}{2})ft$] directed downward. Neglecting the resistance of the medium and assuming no external forces are applied.

Solution: Using Hooke's law $F = KS$, which gives $8 = K\frac{1}{2}$ and so $k = 16lb/ft$. Also, $m = \frac{8}{32}$.

The differential equation is $\frac{d^2x(t)}{dt} = -64x(t), x(0) = (\frac{1}{8},\frac{1}{4},\frac{1}{2},\frac{3}{2};1)$ and $x'(0) = (\frac{1}{2},1,2,\frac{5}{2})$

The solution when $x(t)$ is (i)-gH(Generalized Hukuhara) differentiable and $\frac{dx(t)}{dt}$ is (ii)-gH differentiable or, $x(t)$ is (ii)-gH differentiable and $\frac{dx(t)}{dt}$ is (i)-gH differentiable then we have

$$\frac{d^2x_2(t,\alpha)}{dt^2} = -64x_2(t+\theta,\alpha), \forall\theta \in [-\lambda,0] \tag{5.1}$$

$$\frac{d^2x_1(t,\alpha)}{dt^2} = -64x_1(t+\theta,\alpha), \forall\theta \in [-\lambda,0] \tag{5.2}$$

with, $x_1(0,\alpha) = \frac{1+\alpha}{8}, x_2(0,\alpha) = \frac{3-\alpha}{4}, \frac{dx_1(0,\alpha)}{dt} = \frac{1+\alpha}{2}$ and $\frac{dx_2(0,\alpha)}{dt} = \frac{5-\alpha}{2}$

Solving, we get

$$x_1(t,\alpha) = \frac{1+\alpha}{32}cos8(t+\theta) + \frac{3(1+\alpha)}{32}sin8(t+\theta) \text{ and}$$
$$x_2(t,\alpha) = \frac{7-3\alpha}{32}cos8(t+\theta) + \frac{17-5\alpha}{32}sin8(t+\theta)$$

Table for the solution at t = 1.05, θ= - 1

α	X1	X2
0	0.065291126	0.408360587
0.1	0.071820238	0.393640978
0.2	0.078349351	0.37892137
0.3	0.084878463	0.364201762
0.4	0.091407576	0.349482153
0.5	0.097936688	0.334762545
0.6	0.104465801	0.320042936
0.7	0.110994914	0.305323328
0.8	0.117524026	0.290603719
0.9	0.124053139	0.275884111
1	0.130582251	0.261164503

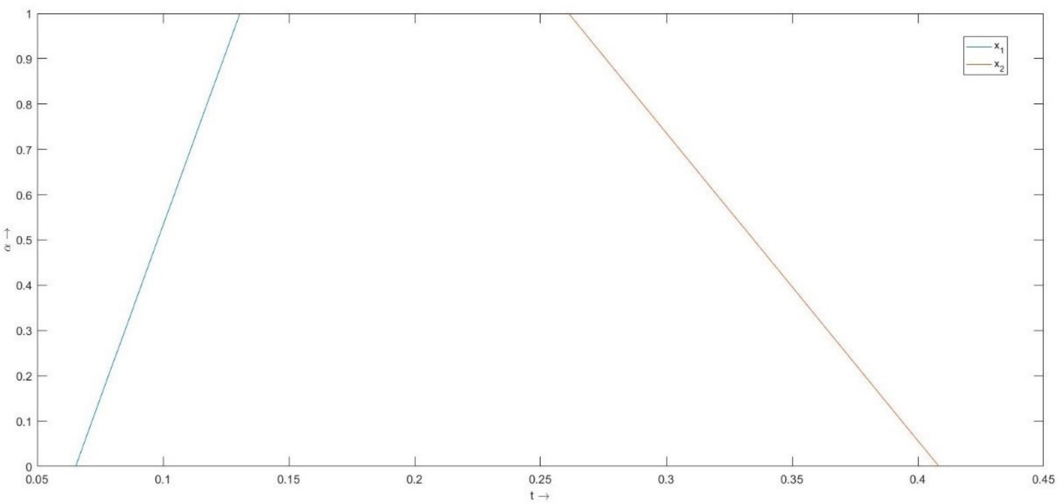

Figure 2: Represents the graphical solution

From the graph, we conclude that as x_1 increases x_2 is decreasing function. Hence, the solution is a strong solution.

6. Conclusions

In this paper, we have proved the controllability of the fuzzy solutions for the second-order impulsive neutral functional differential equation by applying the contraction mapping principle. Further, we can extend the controllability results for the fuzzy inclusions. The numerical solution of the system is also useful for the study of a real-life phenomenon. For instance, we can consider a real-life phenomenon of a friction pendulum bearing specially designed for base isolators used in many heavy structures like bridges, buildings, towers, etc. to reduce the impact of earthquakes. Using the above system, we can also find critical points where the structure becomes unstable or gets damaged.

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Availability of Data and Material

Not applicable

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