Results in Nonlinear Analysis 6 (2023) No. 2, 9–17
https://doi.org/10.31838/rna/2023.06.02.002
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Stability analysis and numerical simulation of fractional model of Leishmaniasis

Mohamed A. Abdoon[a], Abdulgader Bahatheg[b], Mousa Mubarak Albeladi[c], Elsayed E. Elshoubary[d]*

[a]Department of Mathematics, Faculty of Science, Bakht Al-Ruda University, White Nile state, Sudan; [b]Department of Mathematics, Faculty of Science and Arts, Al-baha University, Baljurashi 1988, Saudi Arabia; [c]Department of Mathematics, Faculty of Science and Arts, Jeddah University, Khulais 2014, Saudi Arabia; [d]El-Ahram Institute for Engineering and Technology

Abstract

The disease of leishmaniasis is one that takes some research and study to fully comprehend. As a result, mathematical modeling can be utilized to gain insight into and enhance the accuracy of epidemiological forecasts. A friction model of Leishmaniasis was analyzed using empirical data from Sudan by factoring in the derivatives of Caputo and Atangana-Baleanu. The Caputo and AB derivatives have been subjected to a stability study. A numerical simulation of the suggested ordinary and fractional differential mathematical model follows. The performance was evaluated by calculating its error rating.

Keywords: Mathematical Modelling, Leishmaniosis, Caputo, Atangana–Baleanu derivatives

1. Introduction

Visceral leishmaniasis is a fatal disease transmitted by sandflies. India, Bangladesh, and Nepal have reduced cases of leishmaniasis. Less progress has been made in East Africa, particularly with the continuing epidemic in South Sudan and outbreaks of visceral leishmaniasis. Lack of infrastructure, healthcare personnel, displacement, and malnutrition hamper VL management, diagnostic kits, and medication. However, resistance to pentavalent antimony is a key obstacle that must be overcome before VL may be treated and brought under control. In order to reduce the total amount of time spent

Email addresses: moh.abdoon@gmail.com (Mohamed A. Abdoon), abahathiq@bu.edu.sa (Abdulgader Bahatheg), mhenaidi@uj.edu.sa (Mousa Mubarak Albeladi), smondy1974@gmail.com (Elsayed E. Elshoubary)*

Received March 12, 2023; Accepted April 3, 2023; Online May 8, 2023

receiving treatment for this condition, the first line of treatment, which consisted of sodium stibogluconate for 30 days, has been switched out for a more effective injectable combination regimen that includes SSG and PM, which lasts for only 17 days. Relapse in therapy can occur as a consequence of HIV, tuberculosis, malnutrition, or poor treatment, resulting in the parasite remaining in the blood after the first clinical treatment. Malnutrition is another factor that can contribute to relapse in treatment. Because it is difficult to keep track of active patients in Sudan, the country's rates of VL recurrence are unintentionally and passively increasing because the country accepts VL retreat as a part of the total number of VL admissions [1–7].

Over the course of the last three decades, the fractional calculus has emerged as a topic of increasing interest and significance. The fields of physics, chemical engineering, mathematical biology, and even economics all make use of fractional differential equations and nonlinear equations [8–16].

2. Preliminaries

Definition 1. The Riemann-Liouville fractional integral (RLI) operator of order $\alpha > 0$ for a function $y(\tau)$ is given by [17]:

$$D^{\alpha} y(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} y^n(\tau) d\tau. = I^{n-\alpha} y^n(t),\ t > 0 \tag{1}$$

Definition 2. Caputo derivative of order $0 \le n-1 < \alpha < n$ with the lower limit zero for a function $y(\tau)$ is given by [18]:

$$I^{\alpha} y(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau) d\tau.\ t > 0 \tag{2}$$

Definition 3. For $y \epsilon H^1(0,t), t > 0,\ T > 0,\ \alpha \epsilon (0,1]$. Then the ABC fractional operator [19] $y(t)$ in the Riemann–Liouville is given by

$${}^{AB}_{0}D^{\alpha}_{t} y(t) := \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_0^t y(\tau) E_{\alpha}\left(\frac{\alpha}{1-\alpha} (t-\tau)^{\alpha} \right) d\tau.\ 0 < \alpha < 1 \tag{3}$$

In this expression $B(\alpha)$ satisfies the condition $B(0) = B(1) = 1$.

Definition 4. The Mittag-Leffler function (MLF) is a generalization of the exponential function. This function can be expressed as follows:

$$E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)} \tag{4}$$

3. Anthropological Visceral Leishmaniosis Model with Caputo Derivative

In this Section, the mathematical model of leishmaniasis is a compartmental model with four sub-populations: susceptible, infectious, Recovered, and Recovered and have permanent immunity, for human population and two compartmental for reservoir population Susceptible, infected, in addition to that, we have two compartments for sandflies Susceptible, infected. The human population is the only population in the model that has permanent immunity. The positivity, the number of reproductions, and the equilibrium solutions of the model established in this work have all been determined to be free

of leishmaniasis. Furthermore, the existing cases of leishmaniasis have also been determined along with their respective localities and global stability properties. we obtain the fractional model formulation under Caputo derivative:

$$\mathrm{s}_h(\mathrm{t}) = \frac{\mathrm{S}_H}{\mathrm{N}_H}, \mathrm{i}_h(\mathrm{t}) = \frac{\mathrm{I}_H}{\mathrm{N}_H}, \mathrm{p}_h(\mathrm{t}) = \frac{\mathrm{P}_H}{\mathrm{N}_H}, \mathrm{r}_h(\mathrm{t}) = \frac{\mathrm{R}_H}{\mathrm{N}_H}, \mathrm{s}_r(\mathrm{t}) = \frac{\mathrm{S}_R}{\mathrm{N}_R},$$
$$\mathrm{I}_r(\mathrm{t}) = \frac{\mathrm{I}_R}{\mathrm{N}_R}, \mathrm{s}_V(\mathrm{t}) = \frac{\mathrm{S}_V}{\mathrm{N}_V}, \mathrm{i}_V(\mathrm{t}) = \frac{\mathrm{I}_V}{\mathrm{N}_V}, \mathrm{s}_h(\mathrm{t}) = \frac{\mathrm{S}_H}{\mathrm{N}_H}, m = \frac{N_V}{N_H} \text{ and } N = \frac{N_V}{N_R}$$

The system of differential equations is given by:

$$\begin{cases} {}_0^C D_t^\alpha i_h = abmi_v N_h - \left(\alpha_1 + \delta + \dfrac{A_H}{N_H} - \delta i_h\right) i_h, \\ {}_0^C D_t^\alpha p_h = (1-\sigma)\alpha_1 i_h - \left(\alpha_2 + \beta + \dfrac{A_H}{N_H} - \delta i_h\right) p_h, \\ {}_0^C D_t^\alpha i_r = abni_v s_r - \dfrac{A_H}{N_H} i_r, \\ {}_0^C D_t^\alpha i_v = aci_h S_v + acp_h S_v + aci_r S_v - \dfrac{A_V}{N_V} i_v, \\ {}_0^C D_t^\alpha s_h = \dfrac{A_H}{N_H} - \left[abmi_v + \dfrac{A_H}{N_H} - \delta i_h\right] s_h, \\ {}_0^C D_t^\alpha r_h = \sigma\alpha_1 i_h + (\alpha_2 + \beta) P_h - \left[\dfrac{A_H}{N_H} - \delta i_h\right] r_h, \\ {}_0^C D_t^\alpha S_r = \dfrac{A_R}{N_R} - abni_v s_r - \dfrac{A_H}{N_H} s_r, \\ {}_0^C D_t^\alpha s_v = \dfrac{A_V}{N_V} - \left[aci_h + acP_h + \dfrac{A_V}{N_V}\right] s_v \end{cases} \tag{5}$$

With initial conditions:

$$\mathrm{s}_h(0) = c_1, \mathrm{i}_h(0) = c_2, \mathrm{r}_h(0) = c_3, \mathrm{s}_r(0) = c_4, \mathrm{I}_r(0) = c_5, \mathrm{s}_V(0) = c_6, \quad \mathrm{i}_V(0) = c_7.$$

4. Anthropologic Visceral Leishmaniosis Model with ABC Derivative

We obtain the fractional model formulation under Atangana–Baleanu Caputo derivative:

$$\mathrm{s}_h(\mathrm{t}) = \frac{\mathrm{S}_H}{\mathrm{N}_H}, \mathrm{i}_h(\mathrm{t}) = \frac{\mathrm{I}_H}{\mathrm{N}_H}, \mathrm{p}_h(\mathrm{t}) = \frac{\mathrm{P}_H}{\mathrm{N}_H}, \mathrm{r}_h(\mathrm{t}) = \frac{\mathrm{R}_H}{\mathrm{N}_H}, \mathrm{s}_r(\mathrm{t}) = \frac{\mathrm{S}_R}{\mathrm{N}_R},$$
$$\mathrm{I}_r(\mathrm{t}) = \frac{\mathrm{I}_R}{\mathrm{N}_R}, \mathrm{s}_V(\mathrm{t}) = \frac{\mathrm{S}_V}{\mathrm{N}_V}, \quad \mathrm{i}_V(\mathrm{t}) = \frac{\mathrm{I}_V}{\mathrm{N}_V}, \ \mathrm{s}_h(\mathrm{t}) = \frac{\mathrm{S}_H}{\mathrm{N}_H}, m = \frac{N_V}{N_H} \text{ and } N = \frac{N_V}{N_R}$$

The system of differential equations is given by:

$$\begin{cases} {}_0^{AB}D_t^{\alpha} i_h = abmi_v N_h - \left(\alpha_1 + \delta + \dfrac{A_H}{N_H} - \delta i_h\right) i_h, \\ {}_0^{AB}D_t^{\alpha} p_h = (1-\sigma)\alpha_1 i_h - \left(\alpha_2 + \beta + \dfrac{A_H}{N_H} - \delta i_h\right) p_h, \\ {}_0^{AB}D_t^{\alpha} i_r = abni_v s_r - \dfrac{A_H}{N_H} i_r, \\ {}_0^{AB}D_t^{\alpha} i_v = aci_h S_v + acp_h S_v + aci_r S_v - \dfrac{A_V}{N_V} i_v, \\ {}_0^{AB}D_t^{\alpha} s_h = \dfrac{A_H}{N_H} - \left[abmi_v + \dfrac{A_H}{N_H} - \delta i_h\right] s_h, \\ {}_0^{AB}D_t^{\alpha} r_h = \sigma\alpha_1 i_h + (\alpha_2 + \beta) P_h - \left[\dfrac{A_H}{N_H} - \delta i_h\right] r_h, \\ {}_0^{AB}D_t^{\alpha} S_r = \dfrac{A_R}{N_R} - abni_v s_r - \dfrac{A_H}{N_H} s_r, \\ {}_0^{AB}D_t^{\alpha} s_v = \dfrac{A_V}{N_V} - \left[aci_h + acP_h + \dfrac{A_V}{N_V}\right] s_v \end{cases} \tag{6}$$

With initial conditions:

$$\mathrm{s}_h(0) = c_1, \mathrm{i}_h(0) = c_2, \mathrm{r}_h(0) = c_3, \mathrm{s}_r(0) = c_4, \mathrm{I}_r(0) = c_5, \mathrm{s}_V(0) = c_6, \quad \mathrm{i}_V(0) = c_7 \ .$$

5. Stability Analysis

This section covered aspects including the eigenvalues, Jacobian matrix, and equilibrium points of the kala-azar epidemiological model (1).

5.1. Equilibria

5.2. The Jacobian Matrix for the Model

Here, we discuss this epidemiological model stability. The disease free equilibrium point is given as $\mathbf{E}_1$ = (0,0,0,0,1,0,711.58,1) and the endemic equilibrium points $\mathbf{E}_8$ =(0,0,0,5.1155e8,0,0,0,0).

$$\boldsymbol{J}(E_1) = \begin{bmatrix} -0.031 & 0 & 0 & 0.0157 & 0 & 0 & 0 & 0 \\ 0.002 & -0.933 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4.297e^{-8} & 15907.35 & 0 & 0 & 0 & 0 \\ 0.02 & 0.02 & 0.02 & -3.438e^{-11} & 0 & 0 & 0 & 0 \\ 0.011 & 0 & 0 & -0.0157 & -4.297e^{-8} & 0 & 0 & 0 \\ 0.018 & 0.0933 & 0 & 0 & 0 & -4.297e^{-8} & 0 & 0 \\ 0 & & 0 & -15907.35 & 0 & 0 & -4.297e^{-8} & 0 \\ -0.02 & -0.02 & 0 & 0 & 0 & 0 & 0 & -3.438e^{-11} \end{bmatrix} \tag{7}$$

$$\boldsymbol{J}(E_8)=\begin{bmatrix} -0.031 & 0 & 0 & 0 & 8035456.6 & 0 & 0 & 0 \\ 0.002 & -0.933 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4.297e^{-8} & -0.0006 & 0 & 0 & 11435742383.06 & 0 \\ 0.000002 & 0.00002 & 0.00002 & -3.438e^{-11} & 0 & 0 & 0 & 14.225 \\ 0 & 0 & 0 & 0 & -8035456.6 & 0 & 0 & 0 \\ 0.028 & 0.0933 & 0 & 0 & 0 & -0.00000003 & 0 & 0 \\ 0 & 0 & 0 & 0.0006 & 0 & 0 & -11435742383.06 & 0 \\ -0.00002 & -0.0002 & 0 & 0 & 0 & 0 & 0 & -0.000000027 \end{bmatrix} \tag{8}$$

5.3. The Basic Reproduction Number

The basic reproduction number is a baseline statistic in epidemiology and is represented by R_0, which stands for the predicted value of the secondary infections rate per time unit. Using the equation's fractional model (1), We have fours infected classes, rewrite the system of equation (1) for the susceptible and infected classes in the general form:

$$\frac{dx}{dt} = f(x) - \upsilon(x) \tag{9}$$

Where

$$f(x) = \begin{pmatrix} abmi_\upsilon s_h \\ 0 \\ abmi_\upsilon s_r \\ ac(i_h + p_h + i_r)s_\upsilon) \end{pmatrix}, \text{ and } \upsilon(x) = \begin{pmatrix} (\alpha_1 + \delta + \mu_h)i_h \\ (\alpha_2 + \beta + \mu_h)p_h - (1-\sigma)\alpha_1 i_h \\ \mu_r i_r \\ \mu_\upsilon i_\upsilon \end{pmatrix} \tag{10}$$

Now the Jacobian of $f(x)$ and $\upsilon(x)$ of the disease free equilibrium point is:

$$F = \begin{pmatrix} 0 & 0 & 0 & abm \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & abm \\ ac & ac & ac & 0 \end{pmatrix} \text{and } V = \begin{pmatrix} \alpha_1 + \delta + \mu_h & 0 & 0 & 0 \\ -(1-\sigma)\alpha_1 & \alpha_2 + \beta + \mu_h & 0 & 0 \\ 0 & 0 & \mu_r & 0 \\ 0 & 0 & 0 & \mu_\upsilon \end{pmatrix} \tag{11}$$

we have

$$R_0 = \rho(FV^{-1}) = \sqrt{\frac{ac[\mu_r abm(\alpha_2 + \delta + \mu_h + (1-\sigma)\alpha_1) + abn(\alpha_1 + \delta + \mu_h)(\alpha_2 + \delta + \mu_h)]}{\mu_\upsilon \ \mu_r(\alpha_1 + \delta + \mu_h)(\alpha_2 + \delta + \mu_h)}} \tag{12}$$

Lemma 3.1. The disease-free equilibrium E_0 is locally asymptotically stable if.
$R_0 < 1$ and unstable if $R_0 > 1$.

6. Simulation

In this section, we simulate our model. Table 1 shows the variable values utilized, and Table 2 shows the parameter values used, together with the initial condition, $\mathrm{s}_H(0) = 50, \mathrm{P}_H(0) = 0, \mathrm{i}_H(0) = 1, \mathrm{r}_H(0) = 0, \mathrm{s}_R(0) = 22, \mathrm{I}_R(0) = 1, \mathrm{s}_V(0) = 12, \ \mathrm{i}_V(0) = 1$.

Table 1: Variable values

Variable	Description
$N_H(t)$	Human host population
$N_R(t)$	Reservoir host population
$N_V(t)$	Vector population
$S_H(t)$	Susceptible humans
$P_H(t)$	Recovered and have permanent immunity
$I_H(t)$	Infected humans
$R_H(t)$	Recovery humans
$S_R(t)$	Susceptible reservoir
$I_R(t)$	Infected reservoir
$S_V(t)$	Susceptible sandflies
$I_V(t)$	Infected sandflies

Table 2: Parameter values

Parameter	Description	Value	Source
a	Biting rate of sandflies	0.2856 day^{-1}	[16]
b	Progression rate of VL in sandfly	0.22 day^{-1}	[16]
c	Progression rate of VL in human and reservoir	0.0714 day^{-1}	[27]
A_H	Human recruitment rate	10.1009 day^{-1}	Estimated
A_R	Reservoir recruitment rate	19.7795 day^{-1}	Estimated
A_V	Vector recruitment rate	38858.62 day^{-1}	Estimated
μ_h	Natural mortality rate of humans	4.341e-6 day^{-1}	[2]
μ_r	Natural mortality rate of reservoirs	0.0017 day^{-1}	[1]
μ_v	Natural mortality rate of vectors	0.0668 day^{-1}	[1]
α_1	Treatment rate of VL	0.02	[2]
α_2	PKDL recovery rate without treatment	0.033	[42]
σ	Recovery rate from VL infection after treatment	0.9	[1]
$1-\sigma$	Developing PKDL rate after treatment	0.1	[1]
δ	Death rate due to VL	0.011	[35]
β	PKDL recovery rate after treatment	0.9	[1]

Table 3: Equilibria of the model

E_i	Equilibria
E_1	(0,0,0,0,1,0,711.58,1)
E_2	(32.5436,0.1132,711.5801,22.7897,–29.7254,–1.9314,0,0)
E_3	(–31.1459,–0.0488,711.58,–21.8109,33.9641,–1.7693,0,0)
E_4	(85.0303,0–72.8759,711.5801,59.5451,–82.2121,–71.0578,0,0)
E_5	(2.8182,0.0062,–2.8244,0,0,–1.8244,714.4046,0)
E_6	(2.8182,0.0062,711.5801,252.9373,0,–1.8244,0,0)
E_7	(0,0,0,0, –7.2892e11, 7.2892e11,0,0)
E_8	(0,0,0,5.1155e8,0,0,0,0)

Table 4: The eigenvalues corresponding to matrix $\boldsymbol{J}$ are.

λ_i	Eigenvalues	Stability
λ_1	(17.833,0,0,0,0,0,0, -17.833)	Unstable
λ_2	$(-3.438e^{-11}, -2.8e^{-8}, -2.8e^{-8}, -4.297e^{-8}, -0.31, -0.933, -035456.6, -1.14e^{10})$	Stable

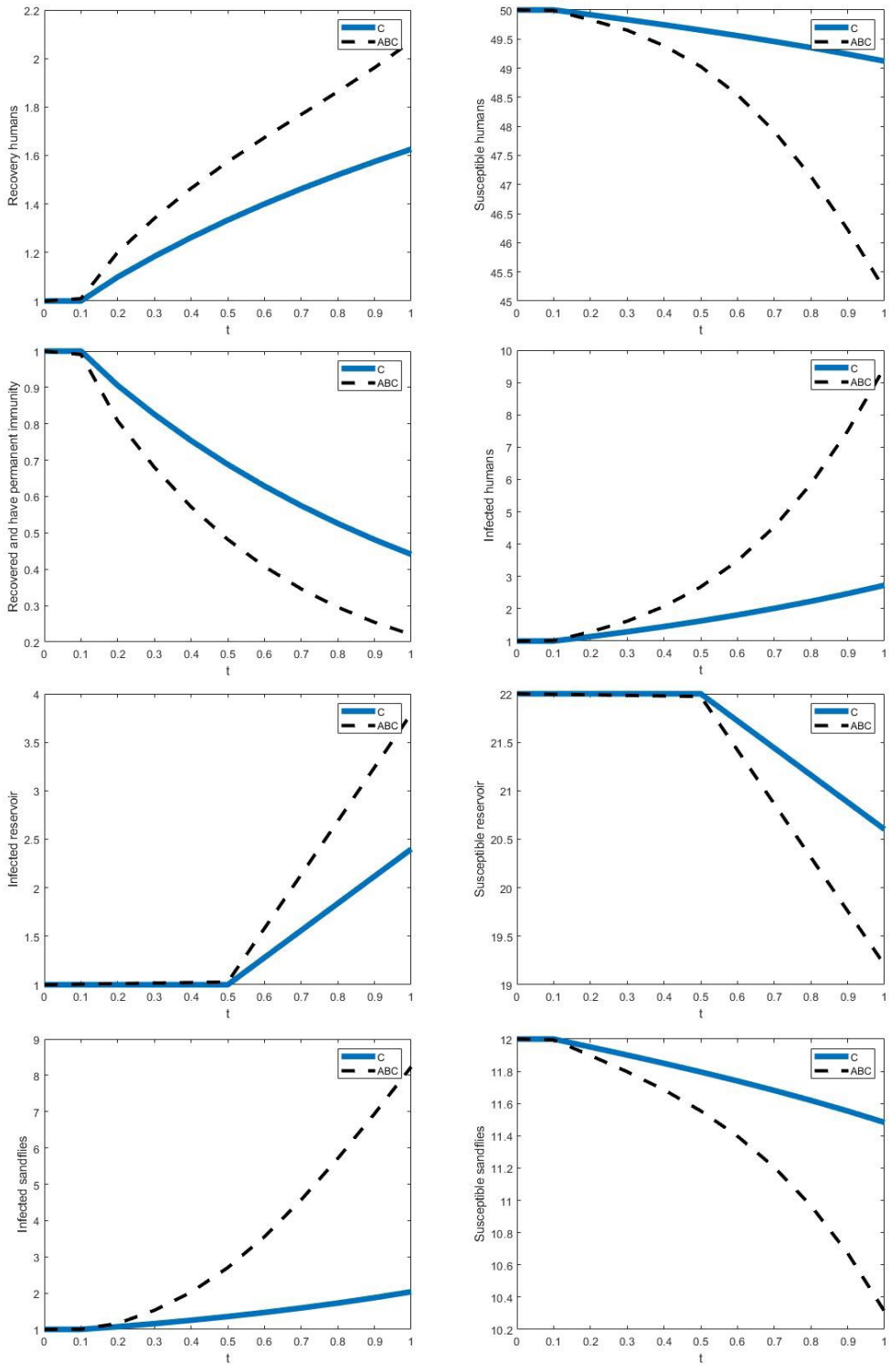

Figure 1: Figures of the systems of fractional orders model for $\alpha = 0.99$.

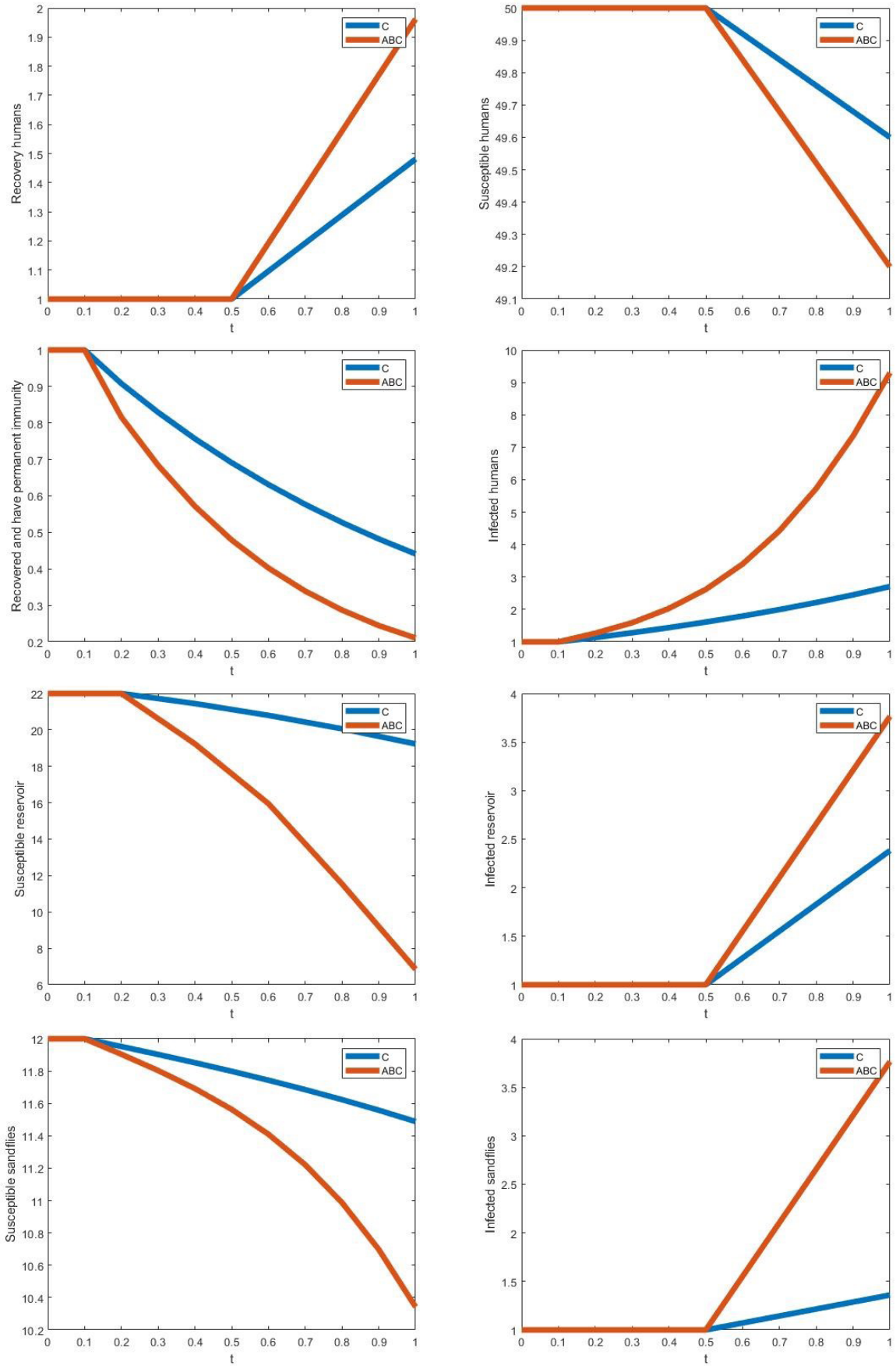

Figure 2: Figures of the fractional orders model for $\alpha = 1$.

7. Conclusion

We used MATLAB to simulate the Caputo derivative and the AB derivative-based fractional model. Furthermore, the model calculations and the corresponding graphs of the fractional derivative provide a comprehensive explanation of Leishmaniosis. The use of fractional derivatives is suggested for a more accurate depiction of the Leishmaniosis pandemic. The AB derivative's superior predictive power may be traced back to the fact that the relevant model has a non-singular kernel.

References

[1] Zijlstra E. E., and el-Hassan A. M., *Leishmaniasis in Sudan. Visceral leishmaniasis*, Trans R Soc Trop Med Hyg, 95 (Suppl 1), (2001 Apr), S27–58. https://doi.org/10.1016/s0035-9203(01)90218-4. PMID: 11370250.

[2] Siddig M., Ghalib H., Shillington D. C., Petersen E. A., and Khidir S., *Visceral leishmaniasis in Sudan. Clinical features*. Tropical and Geographical Medicine, 42 (2), (1990 Apr), 107–112. PMID: 2260205.FATH EKRHMAN

[3] Ahmed M., Abdulslam Abdullah A., Bello I., Hamad S., and Bashir A., *Prevalence of human leishmaniasis in Sudan: A systematic review and meta-analysis*, World J Methodol, 12 (4), (2022 Jul 20), 305–318. https://doi.org/10.5662/wjm.v12.i4.305. PMID: 36159098; PMCID: PMC9350725.EKRHMAN

[4] Om Prakash Singh & Shyam Sundar, *Visceral leishmaniasis elimination in India: progress and the road ahead*, Expert Review of Anti-infective Therapy, 20, (2022), 11, 1381–1388. https://doi.org/10.1080/14787210.2022.2126352 EKRHMAN

[5] Seaman J., Pryce D., Sondorp H. E., Moody A., Bryceson A. D. M., and Davidson R. N., *Epidemic Visceral Leishmaniasis in Sudan: A Randomized Trial of Aminosidine plus Sodium Stibogluconate versus Sodium Stibogluconate Alone*, The Journal of Infectious Diseases, 168 (3), (September 1993), 715–720, https://doi.org/10.1093/infdis/168.3.715

[6] El-Hassan, Ahmed M., et al, *Visceral leishmaniasis in the Sudan: clinical and hematological features*, Annals of Saudi Medicine, 10 (1), (1990), 51–56.

[7] Makau-Barasa, Louise Kathini, et al, *Moving from control to elimination of Visceral Leishmaniasis in East Africa*, Frontiers in Tropical Diseases, (2022), 67.

[8] Leung A. W., *Systems of Nonlinear Partial Differential Equations: Applications to Biology and Engineering*, Springer Science & Business Media, 49, (2013).

[9] Hasan, Faeza L., and Mohamed A. Abdoon, *The generalized (2 + 1) and (3 + 1)-dimensional with advanced analytical wave solutions via computational applications*, International Journal of Nonlinear Analysis and Applications 12 (2), (2021), 1213–1241.

[10] Roubíček T., *Nonlinear Partial Differential Equations with Applications*, Birkhäuser Boston, 153, (2013).

[11] Debnath L., *Nonlinear Partial Differential Equations for Scientists and Engineers*, Springer Science & Business Media, (2012).

[12] Jumarie G., *Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results*, Computers & Mathematics with Applications, 51 (9–10), (2006), 1367–1376.

[13] Mohamed A. Abdoon, Faeza Lafta Hasan, and Nidal E. Taha, *Computational Technique to Study Analytical Solutions to the Fractional Modified KDV-Zakharov-Kuznetsov Equation*, Abstract and Applied Analysis, vol. 2022, Article ID 2162356, 9 pages, (2022). https://doi.org/10.1155/2022/2162356

[14] Miller K. S., and Ross B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, (1993).

[15] Abdoon, M.A., and Hasan, F. L., *Advantages of the differential equations for solving problems in mathematical physics with symbolic computation*, Mathematical Modelling of Engineering Problems, 9 (1), (2022), 268–276. https://doi.org/10.18280/mmep.090133

[16] ELmojtaba, Ibrahim M., Mugisha J. Y. T., and Mohsin H.A. Hashim. *Mathematical analysis of the dynamics of visceral leishmaniasis in the Sudan*. Applied Mathematics and Computation 217 (6), (2010), 2567–2578.

[17] Podlubny I., *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, in: Mathematics in Science and Engineering, 198, (1999).

[18] Caputo M., Fabrizio M., *A new definition of fractional derivative without singular kernel*, Progr. Fract. Differ. Appl, 1 (2), (2015), 73–85.

[19] Atangana A., and Baleanu D., *New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model*, Therm. Sci., 20 (2), (2016), 763–769.