Results in Nonlinear Analysis 6 (2023) No. 1, 59–67
https://doi.org/10.31838/rna/2023.06.01.005
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Dynamics of Lebesgue Quadratic Stochastic Operator with Nonnegative Integers Parameters Generated by 2-Partition

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Abstract

The theory of quadratic stochastic operator (QSO) has been significantly developed since it was introduced in 1920s by Bernstein on population genetics. Over the century, many researchers have studied the behavior of such nonlinear operators by considering different classes of QSO on finite and infinite state spaces. However, all these studies do not comprehensively represent the core problem of QSO; i.e., the trajectory behavior. Recently, a class of QSO called Lebesgue QSO has been introduced and studied. Such an operator got its name based on Lebesgue measure which serves as a probability measure of the QSO. The conditions of the Lebesgue QSO have allowed us to consider the possibility of introducing a new measure for such QSO. This research presents a new class of Lebesgue QSO with nonnegative integers parameters generated by a measurable 2-partition on the continual state space $X = [0,1]$. This research aims to study the trajectory behavior of the QSO by reducing its infinite variables into a mapping of one-dimensional simplex. The behavior of such operators will be investigated computationally and analytically, where the computational results conform to the analytical results. We will apply measure and probability theory as well as functional

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Received January 1, 2023; Accepted February 2, 2023; Online March 9, 2023

analysis to describe the limit behavior of such QSO. We show that for the new class of Lebesgue QSO generated by a 2-partition, one could analyze the behavior of such an operator by describing the existence of fixed points and periodic points of period-2. These results demonstrate that such Lebesgue QSO generated by a measurable 2-partition can be a regular or nonregular transformation dependence of fixed parameters.

Keywords: quadratic stochastic operator; nonlinear operator; Lebesgue measure; infinite state space; trajectory behavior

2010 MSC: 37A05, 37A50

1. Introduction

Let $S(X,F)$ be the set of all probability measures on a measurable space (X,F), where X is a state space and F is a σ-algebra of subsets of X. We then define a family of functions $\{P(x,y,A): x,y \in X, A \in F\}$ on $X \times X \times F$ which satisfy the conditions as follows

(i) $P(x,y,\cdot): F \to [0,1]$ is the probability measure on F, where $P(x,y,\cdot) \in S(X,F)$ for any fixed $x,y \in X$,
(ii) $P(x,y,A)$ is regarded as a function of two variables x and y with fixed $A \in F$ and a measurable function on $(X \times X, F \otimes F)$,
(iii) $P(x,y,A) = P(y,x,A)$.

A quadratic stochastic operator (QSO) is a nonlinear transformation, $V: S(X,F) \to S(X,F)$, defined as follows

$$(V\lambda)(A) = \int_X \int_X P(x,y,A)\, d\lambda(x)\, d\lambda(y), \tag{1}$$

where $\lambda \in S(X,F)$ and $A \in F$ is an arbitrary initial measure and measurable set, respectively.

The idea of QSO was initiated by Bernstein in ([2]) through his work on population genetics as a mathematical discipline. The quadratic stochastic operator theory acts as the main mathematical apparatus which frequently arises in many models of mathematical genetics. The quadratic stochastic operator (QSO) is a mapping of the simplex

$$S^{m-1} = \left\{ \mathbf{x} = (x_1,\ldots,x_m) \in \mathbb{R}^m : x_i \geq 0, \sum_{i=1}^{m} x_i = 1 \right\},$$

into itself such that

$$V: x'_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j,\ k = 1,\ldots,m, \tag{2}$$

where $P_{ij,k}$ are heredity coefficients and

$$P_{ij,k} \geq 0,\ \sum_{k=1}^{m} P_{ij,k} = 1,\ \ i,j,k = 1,\ldots,m. \tag{3}$$

Be mindful that each element $\mathbf{x} \in S^{m-1}$ is a probability distribution on $E = \{1,\ldots,m\}$, where we let E be the set of species types in a population, where $|E| = m$.

Starting from an arbitrary initial state, $\mathbf{x} \in S^{m-1}$, the population evolves to the state $\mathbf{x}' = V\mathbf{x}$, followed by the next state $\mathbf{x}'' = V(\mathbf{x}') = V^2\mathbf{x}$, and so forth. Given $\mathbf{x}^{(0)} \in S^{m-1}$, the trajectory $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$ of QSO in (1) is defined by $\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)})$ where $n = 0,1,2,\ldots$. Thus, one of the most critical problems in mathematical biology which involves the theory of nonlinear operator is the study of the limit behavior of defined trajectories.

Definition 1.1. *A transformation V given by (1) is called a Lebesgue QSO, if $X=[0,1]$ and F is a Borel σ-algebra on* $[0,1]$.

A class of Lebesgue QSO has been introduced and studied in [8], where the authors emphasized the construction of such Lebesgue QSO for both discrete and continuous initial measure. The study of Lebesgue QSO in [8] put focus on the usual Lebesgue measure as the probability measure of the operator and the convergence of such QSO based on the Radon-Nikodym Theorem. They have shown that such operators are regular due to the existence of a strong limit of the sequence, $V^n\lambda$.

Research on the Lebesgue QSO focused on the usual Lebesgue measure, but there has been little work exploring possible forms of probability measure that satisfy the conditions of Lebesgue QSO. In this paper, we are motivated to introduce another family of Lebesgue QSO with a new measure associated with the integral of a function and the usual Lebesgue measure defined on a continual state space. We then describe their trajectory behaviour by considering different values of parameters. Investigating the trajectory behavior of nonlinear operators has been of interest since it was introduced decades ago for the purpose of understanding and describing a specific system. Readers may refer to the study of various nonlinear operators in both in a finite (see [1, 9, 11, 17, 18, 19, 20]) and an infinite state space (see [4, 3, 5, 6, 7, 8, 10, 12, 13, 16, 14, 15]).

2. Lebesgue Quadratic Stochastic Operator with Nonnegative Integers Parameters Generated by 2-Partition

We define a measurable m-partition of the set X such that $\xi=\{A_1,\ldots,A_m\}$ and its corresponding partition on $X\times X$, $\zeta=\{B_{ij}:i,j=1,2,\ldots,m\}$, where $B_{ii}=A_i\times A_i$ for $i=1,2,\ldots,m$ and $B_{ij}=(A_i\times A_j)\cup(A_j\times A_i)$ if $i\neq j$. Hence, we have $B_{ij}=B_{ji}$. we then select a family $\{\mu_{ij}:i,j=1,\ldots,m\}$ of probability measures on (X,F) and define the probability measure $P(x,y,A)$ as follows

$$P(x,y,A)=\mu_{ij}(A) \quad \text{for} \quad (x,y)\in B_{ij}. \tag{4}$$

Then for any $\lambda\in S(X,F)$ and a measurable set $A\in F$ we have

$$\begin{aligned}(V\lambda)(A)&=\int_X\int_X P(x,y,A)d\lambda(x)d\lambda(y)\\&=\sum_{i,j=1}^{m}\int_{A_i}\int_{A_j}\mu_{ij}(A)d\lambda(x)d\lambda(y)\\&=\sum_{i,j=1}^{m}\mu_{ij}(A)\lambda(A_i)\lambda(A_j).\end{aligned}$$

Given that λ is an initial measure and assume $\{V^n\lambda:n=0,1,2,\ldots\}$ is the trajectory of such an initial measure λ, where $V^{n+1}\lambda=V(V^n\lambda)$ for all $n=0,1,2,\ldots$, with $V^0\lambda=\lambda$. Then

$$\begin{aligned}(V^{n+1}\lambda)(A)&=\int_X\int_X P(x,y,A)dV^n\lambda(x)dV^n\lambda(y)\\&=\sum_{i,j=1}^{m}\int_{A_i}\int_{A_j}\mu_{ij}(A)dV^n\lambda(x)dV^n\lambda(y)\\&=\sum_{i,j=1}^{m}\mu_{ij}(A)(V^n\lambda)(A_i)(V^n\lambda)(A_j),\end{aligned}$$

with

$$(V^{n+1}\lambda)(A_k)=\sum_{i,j=1}^{m}\mu_{ij}(A_k)(V^n\lambda)(A_i)(V^n\lambda)(A_j), \tag{5}$$

for $k=1,\ldots,m$.

In measure theory, it is understood that $S(X,F)$ is a weak compact, if X is a compact metric space. For a measurable space (X,F), a sequence μ_n is said to converge strongly to a limit μ if

$$\lim_{n\to\infty} \mu_n(A) = \mu(A),$$

for every set $A \in F$.

Definition 2.1. *A quadratic stochastic operator V is called a regular (weak regular), for any initial measure $\lambda \in S(X,F)$, where the strong limit (respectively weak limit),*

$$\lim_{n\to\infty} V^n(\lambda) = \mu,$$

exists.

Assume that $x_k^{(n)} = (V^n\lambda)(A_k)$ and $P_{ij,k} = \mu_{ij}(A_k)$. Then for $(x_1^{(n)},\ldots,x_m^{(n)}) \in S^{m-1}$, we may rewrite the system of equations as shown in (5) such that

$$(W\mathbf{x})_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \tag{6}$$

for all $k = 1,\ldots,m$, satisfying conditions in (3).

Thus, for a fixed measurable finite partition and selected family of probability measures $\{\mu_{ij} : i,j = 1,\ldots,m\}$, one can approximate QSO V in (1) by a finite-dimensional QSO W in (6).

In this work, we ought to construct another family of Lebesgue QSO with a new form of measure generated by a 2-partition. Then we will describe their trajectory behavior on continual state space $X = [0,1]$.

Let $X = [0,1]$ be a continuous set and F be a Borel σ-algebra of X. On X, we consider a new measure, μ_s, as follows

$$\mu_s((a,b)) = (s+1)\int_a^b x^s dx \tag{7}$$

for any $s \in \{0\} \cup \mathbb{Z}^+$.

Definition 2.2. *A Lebesgue QSO with defined probability measure in (7) is called a homogeneous Lebesgue QSO with nonnegative integer parameters generated by 2-partition if $|s(i,j)| = 1$ for any $i,j = 1,2$.*

Definition 2.3. *A Lebesgue QSO with defined probability measure in (7) is called a nonhomogeneous Lebesgue QSO with nonnegative integer parameters generated by 2-partition if $|s(i,j)| > 1$ for any $i,j = 1,2$.*

Suppose that $\xi = \{A_1, A_2\}$ is a measurable 2-partition of the set X and $\zeta = \{B_{11}, B_{22}, B_{12}\}$ is a corresponding partition of $X \times X = [0,1]\times[0,1]$, where $B_{ii} = A_i \times A_i$ for $i = 1,2$ and $B_{12} = (A_1 \times A_2) \cup (A_2 \times A_1) = B_{21}$. We select a family $\{\mu_{s_{ij}} : i,j = 1,2,\ \text{and}\ s_{ij} \in \{0\} \cup \mathbb{Z}^+\}$ of the new measure as defined in (7) with parameters $s_{11} = s_1$, $s_{22} = s_2$, and $s_{12} = s_3$, and define the probability measure $P(x,y,A)$ as follows

$$P(x,y,A) = \mu_{s_{ij}}(A) \quad \text{for} \quad (x,y) \in B_{ij}, \quad i,j = 1,2, \tag{8}$$

for an arbitrary $A = [a,b] \in F$.

According to $P_{ij,k} = \mu_{s_{ij}}(A_k)$, we have

$$P_{11,1} = \mu_{s_{11}}(A_1), P_{22,1} = \mu_{s_{22}}(A_1), P_{12,1} = \mu_{s_{12}}(A_1),$$

and

$$P_{11,2} = \mu_{s_{11}}(A_2), P_{22,2} = \mu_{s_{22}}(A_2), P_{12,2} = \mu_{s_{12}}(A_2).$$

Now, let any continuous initial measure $\lambda \in S(X,F)$ be a probability measure,

$$A = [a,b] \in F,\ \ A \subset [0,1],\ \ A_1 = [0,\alpha),\ \ A_2 = [\alpha,1],$$

where $\int_{A_1} d\lambda = \lambda(A_1)$, $\int_{A_2} d\lambda = \lambda(A_2)$, and $\alpha \in (0,1)$. Here we examine the following two cases:

(i) Two parameters, where $s_1 = s_2 \neq s_3$,
(ii) Three parameters, where $s_1 \neq s_2 \neq s_3$.

These two cases are necessary to be studied as they represent the nonhomogeneous family of such QSO. Note that if $s_{11} = s_{22} = s_{12}$, then for any $s_{ij} \in \{0\} \cup \mathbb{Z}^+$, the corresponding family of measure is called a homogeneous family. A QSO in (1) generated by a homogeneous family is known as an identity transformation.

For such an identity transformation, it is known that for any initial point λ, one may obtain $V^n(\lambda) = \lambda$ for $n = 0,1,\ldots$. The existence of a strong limit of such sequence suggesting the regularity of such transformation *V*. Hence, it is sensible to consider a nonhomogeneous QSO in this study and investigate its behavior over the defined state space, expecting different trajectory behavior of the QSO from the identity transformation case.

Note that in this paper, we shall consider a nonhomogeneous Lebesgue QSO with measure (7), where $|s(i,j)| = 2,3$ as stated in case (i) and case (ii).

Then, for the case (i), where $s_1 = s_2 \neq s_3$, we have,

$$\begin{aligned}
\lambda'(A) &= \int_0^1\int_0^1 P(x,y,A)d\lambda(x)\ d\lambda(y)\\
&= \int_{A_1}\int_{A_1}\mu_{s_1}(A)d\lambda(x)\ d\lambda(y) + \int_{A_1}\int_{A_2}\mu_{s_3}(A)d\lambda(x)d\lambda(y)\\
&\quad + \int_{A_2}\int_{A_1}\mu_{s_3}(A)d\lambda(x)\ d\lambda(y) + \int_{A_2}\int_{A_2}\mu_{s_2}(A)d\lambda(x)d\lambda(y)\\
&= \mu_{s_1}(A)\ \lambda^2(A_1) + 2\mu_{s_3}(A)\ \lambda(A_1)\ \lambda(A_2) + \mu_{s_2}(A)\ \lambda^2(A_2)\\
&= \mu_{s_1}([a,b])[\lambda^2([0,\alpha)) + \lambda^2([\alpha,1])] + 2\mu_{s_3}([a,b])\ \lambda([0,\alpha))\ \lambda([\alpha,1]).
\end{aligned}$$

By mathematical induction, we may obtain $\lambda''(A)$, such that

$$\lambda''(A) = \mu_{s_1}(A)[(\lambda'(A_1))^2 + (\lambda'(A_2))^2] + 2\mu_{s_3}(A)\ \lambda'(A_1)\ \lambda'(A_2).$$

On the other hand, for the case (ii), where $s_1 \neq s_2 \neq s_3$, we have,

$$\begin{aligned}
\lambda'(A) &= \int_0^1\int_0^1 P(x,y,A)d\lambda(x)\ d\lambda(y)\\
&= \int_{A_1}\int_{A_1}\mu_{s_1}(A)\ d\lambda(x)\ d\lambda(y) + \int_{A_1}\int_{A_2}\mu_{s_3}(A)\ d\lambda(x)\ d\lambda(y)\\
&\quad + \int_{A_2}\int_{A_1}\mu_{s_3}(A)\ d\lambda(x)\ d\lambda(y) + \int_{A_2}\int_{A_2}\mu_{s_2}(A)\ d\lambda(x)\ d\lambda(y)\\
&= \mu_{s_1}(A)\ \lambda^2(A_1) + 2\mu_{s_3}(A)\ \lambda(A_1)\ \lambda(A_2) + \mu_{s_2}(A)\ \lambda^2(A_2)\\
&= \mu_{s_1}([a,b])\lambda^2([0,\alpha)) + 2\mu_{s_3}([a,b])\ \lambda([0,\alpha))\ \lambda([\alpha,1])\\
&\quad + \mu_{s_2}([a,b])\ \lambda^2([\alpha,1]).
\end{aligned}$$

Again, by applying mathematical induction on $\lambda(A)$, we attain the following equation,

$$\lambda''(A) = \mu_{s_1}(A)(\lambda'(A_1))^2 + 2\mu_{s_3}(A)\ \lambda'(A_1)\ \lambda'(A_2) + \mu_{s_2}(A)(\lambda'(A_2))^2.$$

In general, we may refer to the case (ii), where as we apply a mathematical induction on the sequence $\lambda^{(n)}(A)$, we will get

$$\begin{aligned}\lambda^{(n+1)}(A) = \mu_{s_1}(A)(\lambda^{(n)}(A_1))^2 + 2\mu_{s_3}(A)\ \lambda^{(n)}(A_1)\ \lambda^{(n)}(A_2)\\ +\mu_{s_2}(A)(\lambda^{(n)}(A_2))^2,\end{aligned} \tag{9}$$

for $n = 0,1,2,\ldots$.

We may notice that the limit behavior of $\lambda^{(n)}(A_1)$ and $\lambda^{(n)}(A_2)$ determines the limit behavior of (9), where

$$\begin{aligned}\lambda^{(n+1)}(A_1) = \mu_{s_1}(A_1)(\lambda^{(n)}(A_1))^2 + 2\mu_{s_3}(A_1)\ \lambda^{(n)}(A_1)\ \lambda^{(n)}(A_2)\\ +\mu_{s_2}(A_1)(\lambda^{(n)}(A_2))^2,\end{aligned} \tag{10}$$

$$\begin{aligned}\lambda^{(n+1)}(A_2) = \mu_{s_1}(A_2)(\lambda^{(n)}(A_1))^2 + 2\mu_{s_3}(A_2)\ \lambda^{(n)}(A_1)\ \lambda^{(n)}(A_2)\\ +\mu_{s_2}(A_2)(\lambda^{(n)}(A_2))^2,\end{aligned} \tag{11}$$

for $n = 0,1,2,\ldots$.

The recurrent equations in (10) and (11) can be represented as a system of equations, where we denote as a QSO W on a one-dimensional simplex, as follows

$$\begin{aligned}(W\mathbf{x})_1 &= a{x_1}^2 + 2bx_1x_2 + c{x_2}^2,\\ (W\mathbf{x})_2 &= (1-a){x_1}^2 + 2(1-b)x_1x_2 + (1-c){x_2}^2,\end{aligned} \tag{12}$$

such that $a = \mu_{s_1}(A_1)$, $b = \mu_{s_3}(A_1)$, and $c = \mu_{s_2}(A_1)$ are arbitrary coefficients with $0 < a,b,c < 1$. It is apparent that the parameters a, b, and c depend on the 2-partition $\xi = \{A_1, A_2\}$.

3. Trajectory Behavior of Lebesgue QSO with Nonnegative Integers Parameters Generated by 2-Partition

Previously, we have constructed a family of Lebesgue QSO generated by 2-partition with the measure as defined in (7). It is shown that such QSO defined on a continual state space, eventually can be reduced to a one-dimensional setting. In this section, we will describe the trajectory behavior such Lebesgue QSO by analyzing the system of equations in (12) and providing some computational results.

By referring to the system of equations in (12) and the fact that $\sum_{i=1}^{m} x_i = 1$ for $\mathbf{x} = (x_1,\ldots,x_m) \in S^{m-1}$, we know that $x_1 + x_2 = 1$. Hence, it is possible to solve the system of equations in (12) by substitution, where we will obtain a quadratic function as follows

$$x_1' = (a-2b+c){x_1}^2 + 2(a-2b+c)x_1 + c. \tag{13}$$

The study of the one-dimensional QSO which involves the quadratic function in (13) has been solved by Lyubich in [17].

Theorem 3.1. *[17] A fixed point of the transformation in (13) is unique and belongs to* $(0,1)$.

Given that Δ is the discriminant of a quadratic equation,

$$x_1 = (a-2b+c){x_1}^2 + 2(b-c)x_1 + c,$$

where

$$\Delta = 4(1-a)c + (1-2b)^2. \tag{14}$$

Theorem 3.2. *[17] If* $0 < \Delta < 4$*, then a fixed point is attractive, and if* $4 < \Delta < 5$*, then it is repelling.*

Note that we will obtain $0 < \Delta < 4$, when $a = c$ and $4 < \Delta < 5$, when $a \neq c$. Then, the following statement is established.

Theorem 3.3. *[17] If* $0 < \Delta < 4$*, then a one-dimensional QSO (13) is a regular, and if* $4 < \Delta < 5$*, then there is a cycle of second-order. All trajectories tend to this cycle except the stationary trajectory starting with a fixed point.*

To analyze the trajectory behavior of such Lebesgue QSO, we provide some phase diagrams of the iteration of the QSO W in (12) for an arbitrary initial measure and a fixed measurable 2-partition ξ.

The blue colors in the diagram of Figure 1 and Figure 2 correspond to the convergence of the trajectory to a unique fixed point. Meanwhile, the red colors represent the existence of a cycle of

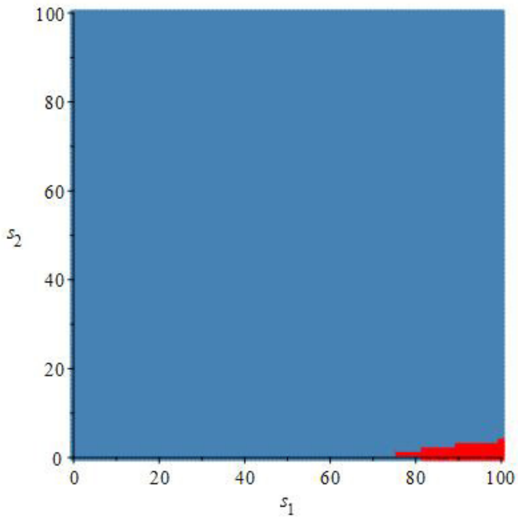

Figure 1: Phase Diagram of QSO W in (12) when $A_1 = [0.05, 0.975)$ and $A_2 = [0, 0.05) \cup [0.975, 1]$ for $s_3 = 0$

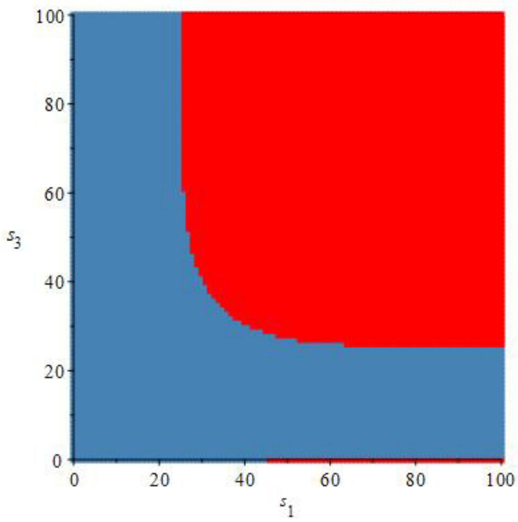

Figure 2: Phase Diagram of QSO W in (12) when $A_1 = [0.025, 0.925)$ and $A_2 = [0, 0.025) \cup [0.925, 1]$ for $s_2 = 1$

second-order. Then, for some values of parameters s_1, s_2, and s_3, the operator in (12) is either regular or nonregular with a second-order cycle. One can observe that for the case (i), where $s_1 = s_2 \neq s_3$, in Figure 1 and Figure 2, such operator is regular, i.e., converges to a unique fixed point, for an arbitrary 2-partition and the small parts of red color represent the condition when $s_1 \neq s_2$. On the other hand, for the case (ii), where $s_1 \neq s_2 \neq s_3$, one can see that for some large values of s_1 and s_3 and small value of s_2, such an operator is nonregular with periodic points of period 2.

We shall provide an example for such Lebesgue QSO.

Example 3.4. *Let $A_1 = [0.025, 0.925)$ and $A_2 = [0, 0.025) \cup [0.925, 1]$ be the measurable 2-partition defined on $X = [0,1]$. Then, we choose $s_1 = 50$, $s_2 = 1$, and $s_3 = 75$. Next, we will obtain the following coefficients,*

$$a = (50+1)\int_{0.025}^{0.925} x^{50}dx = 0.01875980716,$$
$$b = (75+1)\int_{0.025}^{0.925} x^{75}dx = 0.002671601559,$$
$$c = (1+1)\int_{0.025}^{0.925} xdx = 0.8550000000.$$

From this, it is easy to find the value of discriminant of the respective quadratic equation, where $\Delta = 4.345183603$. By solving the quadratic equation, we will get the fixed point of such operator, where $(x_1^*, x_2^*) = (0.3571, 0.6429)$. As mentioned in Theorem 3.3, a QSO W with $4 < \Delta < 5$ will converge to a unique fixed point if $(x_1^{(0)}, x_2^{(0)}) = (x_1^*, x_2^*)$. Otherwise, such an operator will have periodic points of period 2 as follows

$$\mathbf{x}_a^* = (x_{1,a}^*, x_{2,a}^*) = (0.06744085590, 0.9325591441),$$
$$\mathbf{x}_b^* = (x_{1,b}^*, x_{2,b}^*) = (0.7439862790, 0.2560137210).$$

In Figure 3, we show the existence of the periodic points of period 2 when $\mathbf{x}^{(0)} \neq \mathbf{x}^*$ for the QSO W as defined in Example 3.4. This supports our statement in Theorem 3.3 computationally. Hence, the following statement is established.

Proposition 3.5. *A Lebesgue quadratic stochastic operator with the probability measure as defined in (7) generated by 2-partition ξ is either regular or nonregular transformation.*

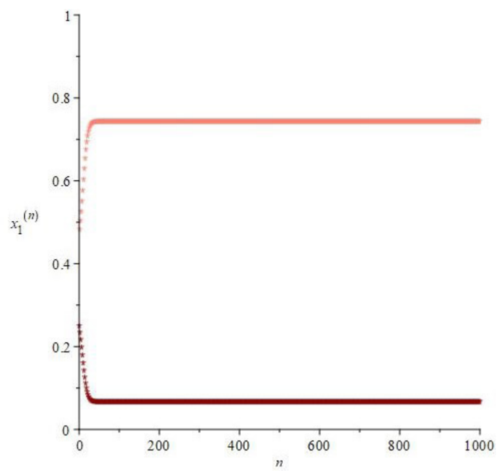

Figure 3: Plot orbit of QSO W in Example 3.4 for an initial measure $\mathbf{x}^{(0)} = (0.25, 075)$

4. Conclusion

In this paper, we have introduced and constructed a new class of Lebesgue QSO generated by a 2-measurable partition with m nonnegative integers parameters, where $m \geq 3$. We have presented two cases that represent the nonhomogeneous family of Lebesgue QSO with a new measure and investigated the limit behavior of such QSO. The main result of this research is given in Proposition 3.5. Such an operator is said to be regular if the trajectory converges to a stable unique fixed point. On the other hand, the existence of a second-order cycle indicates that such operator is nonregular.

Acknowledgments

This research was funded by an FRGS grant by the Ministry of Education Malaysia, project code FRGS/1/2021/STG06/UIAM/02/1 with project ID FRGS21-219-0828 and UMP-IIUM-UiTM Sustainable Research Collaboration 2020 with project ID SRCG20-035-0035

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