



Sinc collocation method for solving linear systems of Fredholm Volterra integro–differential equations of high order with variable coefficients

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Abstract

In this paper, we have implemented Sinc collocation method (SCM) to solve linear systems of higher order Fredholm Volterra integro–differential equations (FVIDEs) with variable coefficients. This method transforms the system FVIDEs into algebraic equations. Two examples are included to illustrate the accuracy and success of the proposed method. We also point out that, as the number of dimensions increases we get satisfactory results, as explained in the figures and tables.

Keywords: Fredholm Volterra IDEs; Sinc collocation; interpolating points

Mathematics Subject Classification (2010): 45B05, 45D05, 45J05, 65L70, 65D15

1. Introduction

Systems of integro–differential equations (IDEs) are widely related to the physical and engineering problems of life such as the ripple of winds in the desert, nano–hydrodynamics, oceanography, modelling, and population growth model [2], [5]. There are many numerical methods in which systems of IDEs. can be solved even if an exact solution can't be found, so knowing the accurate and appropriate method to solve such problems is the most important. Many authors have used different numerical methods to solve these systems such as Darvishi and others [6], by implementing a

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semi-analytical technique to give an approximate solution of systems of FVIDEs and prove the efficiency of the method by adding a higher number of terms. In [7], [13], Keramati et al. have employed Taylor method for solving linear and nonlinear systems of FVIDEs. In addition, Basirat and others [8] suggested another method that combines the Bernstein matrices and the collocation technique to improve and solve the system of FVIDEs. In [9] Attary introduced an approximation to a system of FVIDEs with variable coefficients through the Shannon technique. There are also many methods to approximate the solution of the proposed system, such as Bezier control method [10], Chebyshev polynomial method [11], Modified decomposition method [14], Shifted Chebyshev polynomial method [12], Lagrange method [15]. In [1], Holmaker proved the stability in solving systems of IDEs that describe the construction of liver zones. In this paper, we will use SCM to solve linear systems of higher order FVIDEs with variable coefficients, that is,

$$\sum_{\alpha=0}^r \sum_{i=1}^A h_{ij} U_i^{(\alpha)}(t) = Q_i(t) + \int_0^1 \left(\sum_{i=1}^A \kappa_{ij}(t, \eta) U_i(\eta) \right) d\eta + \int_0^t \left(\sum_{i=1}^A \mu_{ij}(t, \eta) U_i(\eta) \right) d\eta, t \in [0, 1], j = 1, 2, \dots, A, \quad (1)$$

with initial conditions

$$U_i^{(\xi)} = n_{i\xi}, \xi = 0, 1, \dots, \alpha - 1, \quad (2)$$

where $\kappa_{ij}(t, \eta)$, $\mu_{ij}(t, \eta)$, $P_i(t)$ are known functions, and $U_i(t)$ are the unknown functions of equation (1).

This method reduces the solution of system (1) to a solution of an algebraic system by extending the solution $U_i(t)$ into two functions based on the interpolating points and unknown coefficients resulting from the Sinc basis.

We should mention that the time and effort to calculate the solutions, when using this method, were very reasonable.

2. SCM for the Approximate Solution of Linear Systems of FVIDEs

The collocation method is most applied to gain the numerical solution of ordinary differential equations, partial differential equations and integral equations.

If $t \neq 0$, then the sinc function is given by [3]

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}, \quad (3)$$

where the translated sinc function is defined by

$$S(a, h)(t) = \text{sinc}\left(\frac{t - ah}{h}\right), \forall a \in \mathbb{Z}. \quad (4)$$

For the interpolating points $t_l = lh$, the Sinc function is given by [4]:

$$S(a, h)(lh) = \sigma_{al} = \begin{cases} 0, & a \neq l \\ 1, & a = l \end{cases}. \quad (5)$$

If $h \in \mathbb{N}^*$ and $m(t)$ is a function defined on $(-\infty, \infty)$, then the Whittaker expansion of this function is defined by

$$E(m, h)(t) = \sum_{a=-\infty}^{\infty} m(ah) S(a, h)(t). \quad (6)$$

To get an approximation on (v_1, h) , the conformal map

$$w = \delta(z) = \ln\left(\frac{z - v_1}{h - z}\right)$$

is appointed. This map transforms the eye-shaped region

$$B_F = \left\{ z \in C : \left| \arg\left(\frac{z - v_1}{h - z}\right) \right| < \lambda \leq \pi/2 \right\}$$

onto the infinite strip:

$$B_{f_2} = \{w = f_1 + if_2 : |f_2| < \lambda \leq \pi/2\}.$$

The basis functions for $z \in B_F$ on (v_1, h) are chosen as:

$$S(a, h) \circ \delta(t) = \text{sinc}\left(\frac{\delta(t) - ah}{h}\right), \tag{7}$$

where $S(a, h) \circ \delta(t) = S(a, h)(\delta(t))$.

The interpolation points $\{t_l\}$ are then obtained by:

$$t_l = \delta^{-1}(lh) = \frac{v_1 + v_2 e^{lh}}{1 + e^{lh}}, l \in Z. \tag{8}$$

Theorem 2.1. [4] Let $N \in N^*$, $\frac{U}{\delta} \in L_\tau(B_F)$ and h be chosen by

$$h = \left(\frac{\pi\lambda}{\tau M}\right)^{1/2},$$

where $0 < \tau \leq 1, \lambda \leq \frac{\pi}{2}$. In addition to, let $\Psi_{qh}^{(-1)}$ be defined as

$$\Psi_{al}^{(-1)} = \frac{1}{2} + \int_0^{a-l} \text{sinc}(t) d\eta.$$

Then, $\exists c_1 \in N^*$, such that

$$\left| \int_{a_3}^z U(\eta) d\eta - h \sum_{l=-M}^M \mu_{al}^{(-1)} \frac{U(z_l)}{\delta(z_l)} \right| \leq c_1 e^{-(\pi\lambda\tau M)^{1/2}}. \tag{9}$$

Theorem 2.2. [3] If δ is a conformal one to one map of a simply connected domain B_F onto B_F , then

$$\Psi_{al}^{(\alpha)} = \frac{d^\alpha}{d\delta^\alpha} [S(a, l) \circ \delta(t)]_{t=t_l}.$$

Consider the FVIDEs (1) with ICs (2) and let $U_i(t) \in L_\tau(B_F)$.

In the beginning we suppose that the unknown functions $U_i(t)$ are approximated as follows:

$$U_i(t) = P_i(t) + X_i(t), \tag{10}$$

where

$$P_i(t) = \sum_{a=-M}^M p'_a w(t) \text{sinc}\left(\frac{\delta(t) - ah}{h}\right), X_i(t) = \sum_{j=0}^{\alpha} x'_j t^j \tag{11}$$

with p_a are unknown coefficients and $w(t) = t^\alpha (t - 1)^\alpha$.

Apply $\int_0^1 \cdot d\eta$ to (10), we get:

$$\int_0^1 U_t(\eta) d\eta = \int_0^1 P_t(\eta) d\eta + \int_0^1 X_t(\eta) d\eta, \tag{12}$$

and by applying $\int_0^t \cdot d\eta$ to (10), we get:

$$\int_0^t U_t(\eta) d\eta = \int_0^t P_t(\eta) d\eta + \int_0^t X_t(\eta) d\eta. \tag{13}$$

By differentiating both sides of (10) with respect to t , we get

$$U_t^{(\alpha)}(t) = P_t^{(\alpha)}(t) + X_t^{(\alpha)}(t), \tag{14}$$

where

$$P_t^{(\alpha)}(t) = \sum_{a=-M}^M p_a \frac{d^\alpha \left(w(t) \operatorname{sinc} \left(\frac{\delta(t) - ah}{h} \right) \right)}{dt^\alpha}. \tag{15}$$

By substituting (12), (13) and (14) into (1) and find the results of Sinc points $t_l = \frac{e^{lh}}{1 + e^{lh}}$, where $l = -M - 1, \dots, M$, as well as using Theorems (2.1) and (2.2), we will obtain a system of algebraic equations in which the sum of number of t_l 's and number of initial conditions is equal to the sum of number of p_a 's and number of x_j 's, which can be solved to get $\{P_a^l\}_{a=-M}^M$ and $\{x_j^l\}_{j=0}^\alpha$, where the number of t_l 's = $2(M + 1)A$, the number of initial conditions = αA , the number of p_a 's = $(2M + 1)A$, and the number of x_j 's = $(\alpha + 1)A$.

The SCM is illustrated by Algorithm 1

Algorithm 1 Numerical realization Using SCM

1) Input

- M, α, A, τ, ξ
- $Q_t(t)$
- $\kappa_{\iota j}(t), \mu_{\iota j}(t)$ for $\iota = 1, \dots, A; j = 1, \dots, A$
- Initial condition $U_t(0)$
- Initial condition $U_t^{(\xi)}(0)$ for $\xi = 1, \dots, \alpha - 1$

2) Define

- Sinc function $S(a, h, t)$
- $X_t(t) = \sum_{j=0}^\alpha x_{\iota j} t^j$
- $P_t(t) = \sum_{j=-M}^M p_{\iota j} S(m, l, t)$

3) Replace $U_t(t) = P_t(t) + X_t(t)$

4) Solve

- interpolating point $t_l, l = -M - 1, -M, \dots, M$
- $U_t(t_l) = Q_t(t_l) + \int_0^1 \sum_{\iota=1}^A \kappa_{\iota j}(t_l, \eta) U_t(\eta) d\eta + \int_0^{t_l} \sum_{\iota=1}^A k_{\iota j}(t_l, \eta) U_t(\eta) d\eta$
- $U_t(0) = \tilde{U}_t(0)$
- $U_t^{(\xi)}(0) = \tilde{U}_t^\xi(0)$

5) Resolving the algebraic equations to obtain $p_{\iota j}$ and $x_{\iota j}$

3. Numerical Examples

Now we will present an approximate solution to some systems of FVIDEs with variable coefficients using SCM.

We took $\lambda = \pi/2$, and $\tau = 1/2$ for all examples

Example 3.1. Consider the following FVIDEs

$$\begin{aligned}
 4U_1'(x) - xU_2(x) + U_3(x) &= Q_1(x) + \int_0^1 [(x^3 - t^2)U_1(t) + (4t - 3x)U_2(t) + 26xU_3(t)]dt \\
 &\quad + \int_0^x [8U_1(t) + x^5U_2(t) + 2U_3(t)]dt, \\
 (x^2)U_2'(x) + 3U_1(x) &= Q_2(x) + \int_0^1 [(13t - 4x)U_1(t) + (\tanh x)U_2(t) + 6t^3U_3(t)]dt \\
 &\quad + \int_0^x [(4x^2 + t)U_1(t) + U_2(t) + 7xU_3(t)]dt, \\
 (\sin x)U_3'(x) - U_2(x) - 2U_1(x) &= Q_3(x) \\
 &\quad + \int_0^1 [(x^5 - 5t^2)U_1(t) + (3 - x^2)U_2(t) + (3 + 4t)U_3(t)]dt \\
 &\quad + \int_0^x [(3x^4)U_1(t) + (4t^4)U_2(t) + U_3(t)]dt,
 \end{aligned} \tag{16}$$

where

$$\begin{aligned}
 Q_1(x) &= \frac{125}{3} - 37x + 25x^2 - \frac{9}{2}x^3 - 4x^4 + 22x^6 - x^8, \\
 Q_2(x) &= \frac{-15}{2} + 28x - \frac{x^2}{2} + \frac{21}{2}x^3 - \frac{2}{5}x^5 - 2x^6 + 21\tanh x, \\
 Q_3(x) &= \frac{535}{6} - \frac{47}{2}x^2 - 4x^3 + \frac{131}{10}x^5 - \frac{12}{7}x^7 - \frac{3}{2}x^8 - \sin x
 \end{aligned}$$

with ICs:

$$U_1(0) = 1, U_2(0) = -22, U_3(0) = 0.$$

The analytical solution of (16) is

$$U_1(x) = 2x^3 + 1, U_2(x) = 3x^2 - 22, U_3(x) = -x.$$

Now we apply Algorithm 1 to solve the system (16).

Tables 1 and 2 contain the solution by SCM. Tables 3 and 4 show the error for SCM with $M = 3, M = 4$. Figure 1.(A,B,C) shows the solution by SCM with $M = 3$. Figure 2.(A,B,C) shows the solution by SCM with $M = 4$. Figure 3 shows the error by SCM.

Example 3.2. Consider the following FVIDEs

$$\begin{aligned}
 e^x U_1'''(x) + x^2 U_1''(x) + U_2''(x) &= Q_1(x) + \int_0^1 \left[\left(\frac{5}{2}t \right) U_1(t) + \left(\frac{7}{2}x \right) U_2(t) \right] dt \\
 &\quad + \int_0^x [(5t - x)U_1(t) + (5t + x)U_2(t)]dt, \\
 e^x U_2'''(x) + x^3 U_2''(x) + 4U_1'(x) &= Q_2(x) + \int_0^1 \left[\left(\frac{7}{3}t \right) U_1(t) + \left(\frac{7}{3}x \right) U_2(t) \right] dt \\
 &\quad + \int_0^x [(x - 7t)U_1(t) + (x + 7t)U_2(t)]dt,
 \end{aligned} \tag{17}$$

Table 1: Solution of U_1 and U_2 for system (16).

x	U_{1exact}	U_{1app}		U_{2exact}	U_{2app}	
		M = 3	M = 4		M = 3	M = 4
0	1	1	1	-22	-22	-22
0.1	1.002	1.002	1.002	-21.97	-21.97	-21.97
0.2	1.016	1.016	1.016	-21.88	-21.88	-21.88
0.3	1.054	1.054	1.054	-21.73	-21.73	-21.73
0.4	1.128	1.128	1.128	-21.52	-21.52	-21.52
0.5	1.25	1.25	1.25	-21.25	-21.25	-21.25
0.6	1.432	1.432	1.432	-20.92	-20.92	-20.92
0.7	1.686	1.686	1.686	-20.53	-20.53	-20.53
0.8	2.024	2.024	2.024	-20.08	-20.08	-20.08
0.9	2.458	2.458	2.458	-19.57	-19.57	-19.57

Table 2: Solution of U_3 for system (16).

x	U_{3exact}	U_{3app}	
		M = 3	M = 4
0	0	0	0
0.1	-0.1	-0.1	-0.1
0.2	-0.2	-0.2	-0.2
0.3	-0.3	-0.3	-0.3
0.4	-0.4	-0.4	-0.4
0.5	-0.5	-0.499999	-0.5
0.6	-0.6	-0.599999	-0.6
0.7	-0.7	-0.7	-0.7
0.8	-0.8	-0.8	-0.8
0.9	-0.9	-0.9	-0.9

Table 3: Error of U_1 and U_2 for system (16).

x	$ U_1 - U_{1app} $		$ U_2 - U_{2app} $	
	M = 3	M = 4	M = 3	M = 4
0	0	0	0	0
0.1	3.3411e-9	1.45811e-10	2.54549e-7	8.68354e-10
0.2	7.75286e-9	5.63727e-10	5.19588e-7	2.07003e-9
0.3	2.16689e-8	1.34166e-9	2.16164e-7	5.06915e-10
0.4	9.434e-8	2.45523e-9	5.25623e-7	4.31017e-9
0.5	1.46346e-7	3.18166e-9	1.00497e-6	4.80668e-9
0.6	1.00975e-7	2.77823e-9	7.78405e-7	1.90317e-9
0.7	3.11504e-8	1.5904e-9	9.71665e-8	1.03456e-9
0.8	1.4162e-7	6.78192e-10	4.06849e-7	1.86478e-9
0.9	1.56406e-7	2.31494e-10	4.67487e-7	1.64211e-9

Table 4: Error of U_3 for system (16).

x	$ U_3 - U_{3app} $	
	$M = 3$	$M = 4$
0	0	0
0.1	$2.40681e-7$	$8.9125e-10$
0.2	$4.79611e-7$	$2.83849e-9$
0.3	$2.31625e-7$	$6.76723e-9$
0.4	$4.01656e-7$	$1.33012e-8$
0.5	$8.27506e-7$	$1.85829e-8$
0.6	$6.57299e-7$	$1.87364e-8$
0.7	$8.87771e-8$	$1.47644e-8$
0.8	$3.45183e-7$	$9.88178e-9$
0.9	$4.08934e-7$	$4.51731e-9$

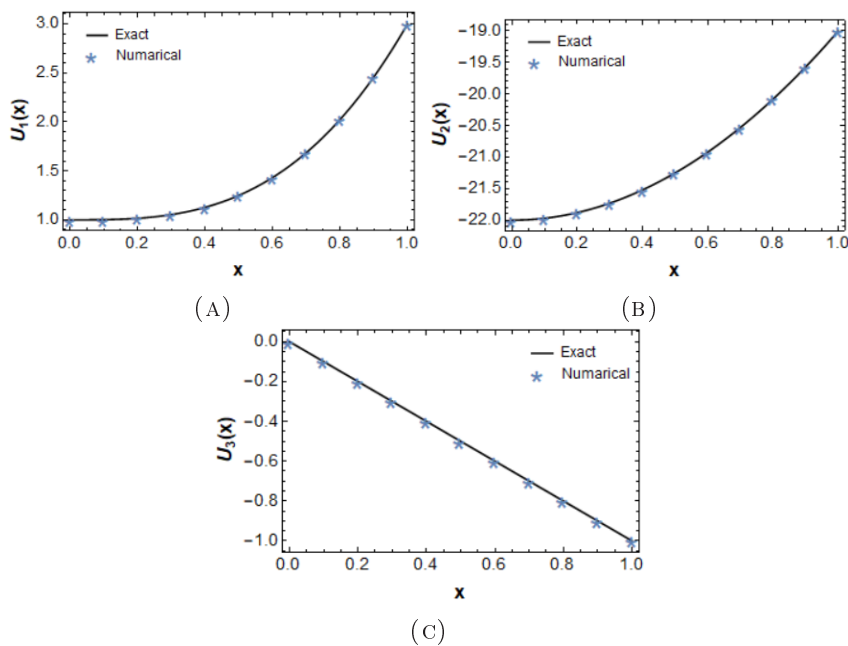


Figure 1: Solution of system (16) by SCM ($M = 3$).

where

$$Q_1(x) = \frac{91}{12} - \frac{91}{6}x + 3x^2 - 4x^3 - 10x^4,$$

$$Q_2(x) = -\frac{295}{18} + \frac{197}{9}x - 2x^2 - \frac{32}{3}x^3 - \frac{8}{3}x^4$$

with ICs:

$$U_1(0) = U_2(0) = 1, U_1'(0) = -4, U_2'(0) = 4, U_1''(0) = U_2''(0) = 8.$$

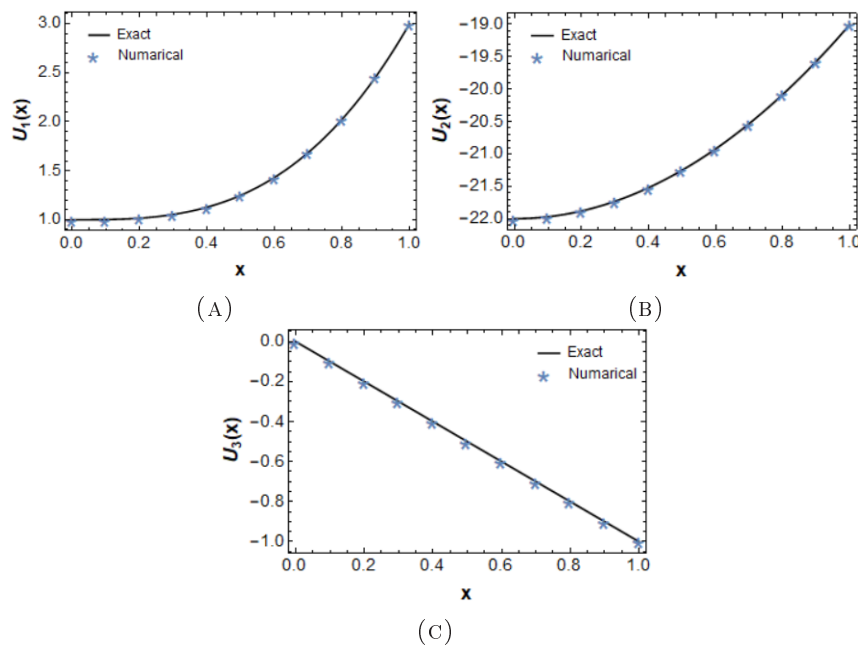


Figure 2: Solution of system (16) by SCM ($M = 4$).

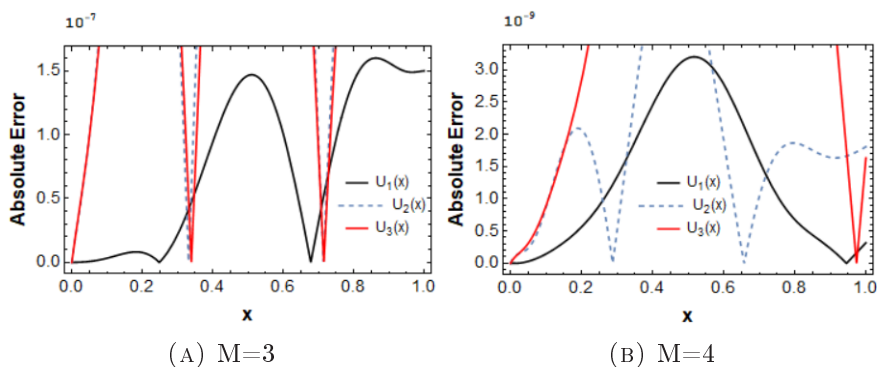


Figure 3: Error by SCM for system (16).

Table 5: Solution of system (17).

x	U_{1exact}	U_{1app}		U_{2exact}	U_{2app}	
		M = 5	M = 6		M = 5	M = 6
0	1	1	1	1	1	1
0.1	0.64	0.640119	0.64	1.44	1.43981	1.44
0.2	0.36	0.360494	0.36	1.96	1.95924	1.96
0.3	0.16	0.161163	0.16	2.56	2.55829	2.56
0.4	0.04	0.0421568	0.04	3.24	3.23694	3.24
0.5	0	0.00350293	-3.63282e-8	4	3.99516	4
0.6	0.04	0.0452234	0.04	4.84	4.83293	4.84
0.7	0.16	0.167336	0.16	5.76	5.7502	5.76
0.8	0.36	0.369854	0.36	6.76	6.74693	6.76
0.9	0.64	0.652787	0.64	7.84	7.82308	7.84

The analytical solution of (17) is

$$U_1(x) = (1 - 2x)^2, U_2(x) = (1 + 2x)^2.$$

Now we apply Algorithm 1 to solve the system (17).

Table 5 contains the solution by SCM. Table 6 shows the error for SCM with $M = 5, M = 6$. Figure 4.(A,B) shows the solution by SCM with $M = 5$. Figure 5.(A,B) shows the solution by SCM with $M = 6$. Figure 6 shows the error by SCM.

Table 6: Error of system (17).

x	$ U_1 - U_{1app} $		$ U_2 - U_{2app} $	
	M = 5	M = 6	M = 5	M = 6
0	0	0	0	0
0.1	1.18927e-4	4.93331e-10	1.90706e-4	3.60051e-10
0.2	4.94069e-4	5.8574e-10	7.58348e-4	1.12957e-9
0.3	1.16277e-3	1.1695e-8	1.70992e-3	1.19834e-8
0.4	2.15679e-3	2.94792e-8	3.06187e-3	2.89078e-8
0.5	3.50293e-3	3.63282e-8	4.839e-3	3.5178e-8
0.6	5.22339e-3	1.87501e-8	7.07327e-3	1.75677e-8
0.7	7.33596e-3	1.39048e-8	9.80266e-3	1.55183e-8
0.8	9.85403e-3	3.40151e-8	1.30701e-2	3.77311e-8
0.9	1.27866e-2	2.83699e-8	1.69224e-2	3.59158e-8

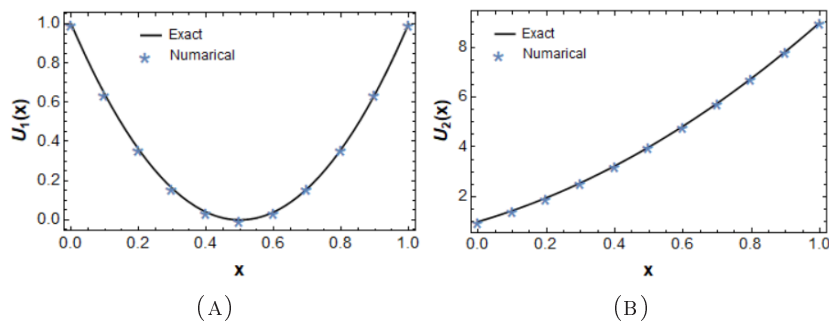


Figure 4: Solution of system (17) by SCM ($M = 5$).

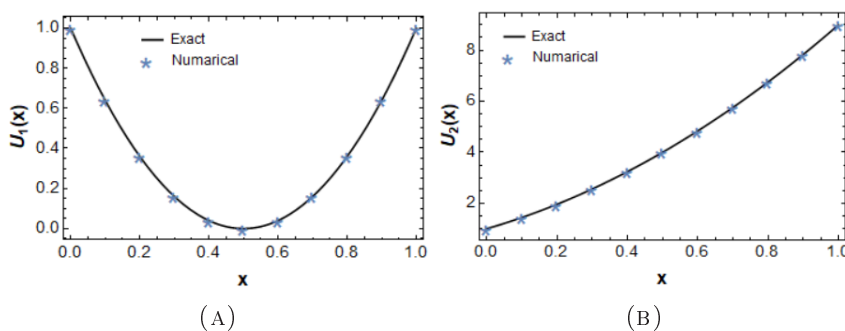


Figure 5: Solution of system (17) by SCM ($M = 6$).

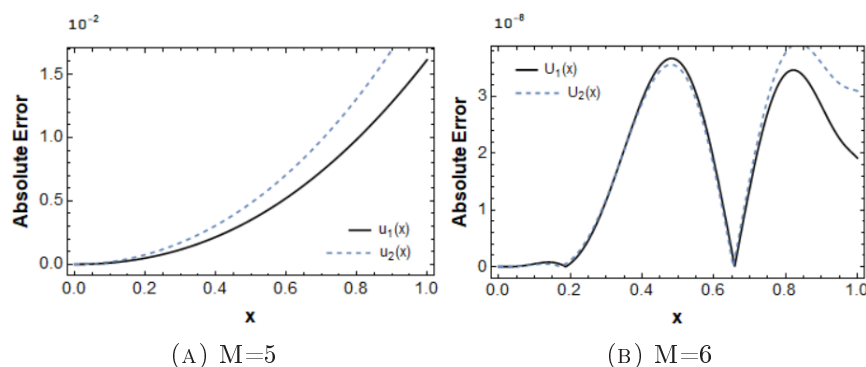


Figure 6: Error by SCM for system (17).

4. Conclusion

In this article, an accurate algorithm (Sinc Algorithm) was constructed to solve linear systems of Fredholm Volterra integro–differential equations (FVIDEs), also we noticed that the results obtained using the proposed algorithm were very satisfactory compared with analytical solutions. In addition, we observed that when the dimension is increased, the absolute error is reduced and gives results that achieve high accuracy with analytical solutions.

References

- [1] K. Holmaker, *Global Asymptotic Stability For A stationary Solution of a system of Integro–Differential Equations Describing the Formation of Liver Zones*, Siam j. Math. Anal., 1993, 24(1): 116–128.
- [2] T. Bo, L. Xie, X. Zheng, *Numerical Approach to Wind Ripple in Desert*, International Journal of Nonlinear Sciences and Simulation, 2007, 8(2): 223–228.
- [3] A.Issa, N. Qatanani, A. Daraghmeh, *Approximation Techniques for Solving Linear Systems of Volterra Integro– Differential Equations*, Journal of Applied Mathematics, 2020.
- [4] Mohsen, M. El–Gamel, *A Sinc–Collocation Method for the Linear Fredholm Integro–Differential Equations*, Zeitschrift für angewandte Mathematik and Physik, 2007, 58(3): 380–390.
- [5] J. Abdul Jerri, *Introduction to Integral Equations with Applications*, New York: Wiley, 1999.
- [6] S. Kheybari, M. T. Darvishi, A. M. Wazwaz, *A semi–Analytical Algorithm to solve Systems of Integro–Differential Equations under Mixed Boundary Conditions*, Journal of Computational and Applied Mathematics, 2017, 317: 72–89.
- [7] Y. Jafarzadeh, B. Keramati, *Numerical Method for A system of Integro–Differential Equations And Convergence Analysis by Taylor Collocation*, Ain Shams Engineering Journal, 2018, 9(4): 1433–1438.
- [8] K. Maleknejad, B. Basirat, E. Hashemizadeh, *A Bernstein Operational Matrix Approach for Solving A system of High order Linear Volterra–Fredholm Integro–Differential Equations*, Mathematical and Computer Modelling, 2012, 55(3–4): 1363–1372.
- [9] K. Maleknejad, M. Attary, *A Chebyshev Collocation Method for the Solution of Higher–order Fredholm–Volterra Integro–Differential Equations System*, Scientific Bulletin, Universitatea Politehnica din Bucuresti Series A, 2012, 74(4): 17–28.
- [10] P. V. Ayoo, R. E. Amobeda, O. Akinwale, *Solution of Systems of Disjoint Fredholm–Volterra Integro–Differential Equations using Bezier Control Points*, Science World Journal, 15(4): 2020, 35–40.
- [11] Akyüz–Daşcıoğlu, M. Sezer, *Chebyshev Polynomial Solutions of Systems of Higher–order Linear Fredholm–Volterra Integro–Differential Equations*, Journal of the Franklin Institute, 2005, 342(6): 688–701.
- [12] E. Hesameddini, M. Riahi, *Shifted Chebyshev Polynomial Method for Solving Systems of Linear And Nonlinear Fredholm–Volterra Integro–Differential Equations*, Journal of Mathematical Extension, 2018, 12(1): 55–79.
- [13] M. Gülsu, M. Sezer, *Taylor Collocation Method for Solution of Systems of High–order Linear Fredholm–Volterra Integro–Differential Equations*, International Journal of Computer Mathematics, 2006, 83(4): 429–448.
- [14] A. Jalal, N. A. Sleman, A.I. Amen, *Numerical Methods for Solving the System of Volterra–Fredholm Integro– Differential Equations*, ZANCO Journal of Pure and Applied Sciences, 2019, 31(2): 25–30.
- [15] Y. Jafarzadeh, B. Keramati, *Numerical Method for A system of Integro–Differential Equations by Lagrange Interpolation*, Asian–European Journal of Mathematics, 2016, 9(4): 1650077.