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Some exact solutions of a new (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation

Wei Ni, Yuhao Tang, Yezhou Li*

School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China.

Abstract

By using symbolic computation and complex method, we present abundant exact solutions of a new (3+1)-dimensional Boiti-Leon-Manna-Pemp-inelli equation, which contain the periodic solitary wave solutions, the interaction solutions between lump wave and solitary waves and the meromorphic solutions.

Keywords: Periodic Solitary Wave; Lump Wave; Meromorphic Solutions; Complex Method

2010 MSC: 33F10, 35C08, 30D35

1. Introduction

It is well acknowledged that the exploration of many significant natural and engineering problems can be intimately related to the study of nonlinear partial differential equations (NLPDEs). For example, KdV equation [1] is regarded as a dynamic model for long wave propagating in a channel, and Laplace equation [2] is applied to the research about the steady state heat conduction problem. For the study of incompressible fluid, i.e. fluids with negligibly density change during the process of flowing, Darvishi [3] introduced the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli (3D-BLMP) equation

$$u_{yt} + u_{zt} + u_{xxxy} + u_{xxxz} - 3u_x(u_{xy} + u_{xz}) - 3u_{xx}(u_y + u_z) = 0.$$

**Email addresse:* yezhouli2019@outlook.com (Yezhou Li)

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Since then, the exact solutions to this equation have been discovered through the extensive work of numerous scholars. For example, Ma [4] used Wronskian technology to get the Wronskian Determinant solutions, Liu considered more abundant soliton structures [5], and also obtained the three wave solutions [6]. Their methods of solving equations mainly included Lie symmetry method [7], [8], (G'/G)-expansion method [9], Bäcklund transformation method [10], etc. Fluid properties such as continuity, fluidity etc. can be uncovered and explored by analyzing their solutions. The laws of nature can be explained more effectively by using NLPDEs.

Considering a more complicated case, Wazwaz [11] added another possible derivative u_x to the derivative term $u_y + u_z$ of the 3D-BLMP equation and got a new equation:

$$(u_x + u_y + u_z)_t + (u_x + u_y + u_z)_{xxx} + (u_x(u_x + u_y + u_z))_x = 0, \tag{1}$$

where $u = u(x, y, z, t)$. Furthermore, he used Painlevé analysis to obtain the compatibility condition and confirm its integrability. This work enlarged the category of integrable equations with time-dependent coefficients. In the same article, the bilinear form and multiple soliton solutions of Eq. (1) were given, which greatly promote the study of it. Some breather wave, lump-type solutions and interaction solutions of Eq. (1) were obtained in Yuan's [12] and Liu's [13] papers respectively. After a precise analysis of the mathematical model, Khalid K. Ali [14] used the tanh function method and sine Gordon expansion method to find some new soliton solutions with satisfactory accuracy. Chen [15] found a variety of kink solutions using methods such as the three wave method. In particular, they also obtained the kink-shaped solitary wave solutions with a tail, which can be used to explain some physical phenomena. Recently, Mehwish Rani et al. [16] have obtained the kink and periodic rational solutions of the Eq. (1) through an improved $\tanh(\frac{\varphi}{2})$-expansion method, and have proved the effectiveness of the method.

According to the applicable field of Eq. (1), periodic solitary wave solutions can help to understand the wave fluctuation phenomenon in shallow water, which can occur in both rivers and seas. Moreover, it is well known that lump waves can be regarded as limiting forms of solitons and can propagate higher propagation energy [17], [18], [19]. So in the context of ocean, the appearance of lump waves may bring some disasters. Therefore, the study of lump waves is helpful to better understand and predict the possible extreme conditions in nonlinear systems. In regards to the NLPDEs in the complex plane, the meromorphic solutions are difficult to obtain because we have to consider the factor of singularities. The behavior of solutions near singularities is also complex, so it is very meaningful to investigate meromorphic solutions.

The purpose of this paper is to study Eq. (1) and get some new meromorphic solutions, periodic solitary wave solutions and interaction solutions between lump wave and solitary waves of it. The properties of them are explained by plotting figures.

This paper is organized as follows: Section 2 derive the new periodic solitary wave solutions and interaction solutions between lump wave and solitary waves with the homoclinic test approach, Section 3 persents the exact meromorphic solutions obtained by complex method, and the Section 4 summarises the main conclusions.

2. Soliton Solutions

2.1. Periodic Solitary Wave Solutions

The transformation of Eq. (1) is applied as:

$$u = -2[\ln f(x, y, z, t)]_x,$$

then it transforms into the bilinear form

$$[D_x^4 + D_y D_x^3 + D_z D_x^3 + D_t D_x + D_t D_y + D_t D_z] f \cdot f = 0. \tag{2}$$

By the definition of D-operators, Eq. (2) can be rewritten as follows:

$$\begin{aligned}&2f(f_{xxxx}+f_{xxxy}+f_{xxxz})+f_{xxx}(-8f_x-2f_y-2f_z)-6f_{xxy}f_x-6f_{xxz}f_x\\&+6f_{xx}^2+f_{xx}(6f_{xy}+6f_{xz})+2f(f_{xt}+f_{yt}+f_{zt})-2f_t(f_x+f_y+f_z)=0.\end{aligned}\tag{3}$$

Supposing Eq. (3) has the solutions with the following form:

$$f(x,y,z,t)=e^{-\xi_1}+\delta_1\cos(\xi_2)+\delta_2 e^{\xi_1},\tag{4}$$

where $\xi_i=a_ix+b_iy+c_iz+d_it, a_i, b_i, c_i, d_i$ and δ_i are undetermined constant. Substituting Eq. (4) into Eq. (3) and letting the corresponding coefficients of $e^{-\xi_1}\cos(\xi_2), e^{-\xi_1}\sin(\xi_2), e^{\xi_1}\cos(\xi_2)$, $e^{\xi_1}\sin(\xi_2), \sin^2(\xi_2), \cos^2(\xi_2)$ and constant term be zero. The following cases are obtained.

Case 1

$$\begin{aligned}&a_1=a_1, b_1=-a_1-c_1, c_1=c_1, d_1=-4a_1^3,\\&a_2=-b_2-c_2, b_2=b_2, c_2=c_2, d_2=-4(b_2+c_2)^3, \delta_1=\delta_1, \delta_2=\delta_2,\end{aligned}$$

where $a_1, b_2, c_1, c_2, \delta_1, \delta_2$ are arbitrary real constants.

Bring these values into Eq. (4) and get the solution of Eq. (1) as follows:

$$u_1(x,y,z,t)=-\frac{2(a_1\delta_2e^{\xi_1}-a_2\delta_1\sin(\xi_2)-e^{-\xi_1}a_1)}{\delta_2e^{\xi_1}+\delta_1\cos(\xi_2)+e^{-\xi_1}},$$

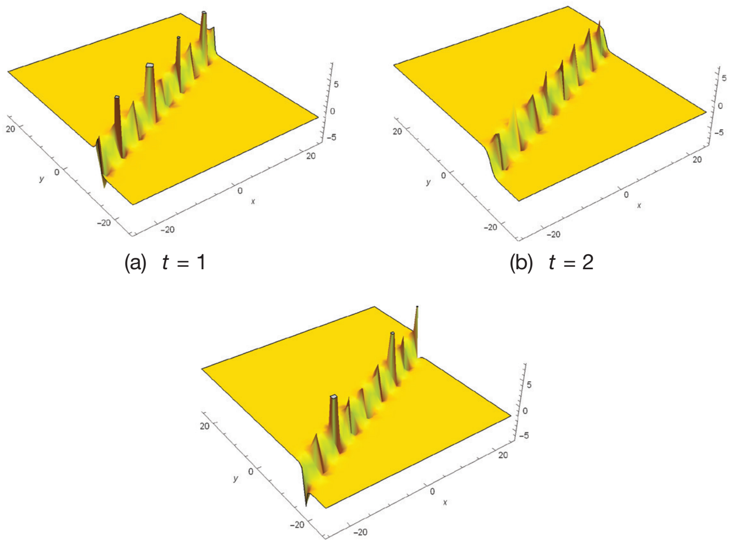
where $\xi_1=a_1x+(-a_1-c_1)y+c_1z-4a_1^3t, \xi_2=(-b_2-c_2)x+b_2y+c_2z-4(b_2+c_2)^3t$.

The dynamical behavior to $u_1(x,y,z,t)$ is demonstrated in Figure 1.

Case 2

$$\begin{aligned}&a_1=0, b_1=b_1, c_1=c_1, d_1=0,\\&a_2=a_2, b_2=-a_2-c_2, c_2=c_2, d_2=a_2^3, \delta_1=\delta_1, \delta_2=\delta_2,\end{aligned}$$

where $a_2, b_1, c_1, c_2, \delta_1, \delta_2$ are arbitrary real constants.



(a) $t=1$ (b) $t=2$ (c) $t=3$

Figure 1: u_1 with $a_1=c_1=c_2=\delta_2=1, b_2=0, \delta_1=z=2$

Substituting these values into Eq. (4), then leads to

$$u_2(x,y,z,t) = \frac{2a_2\delta_1 \sin(\xi_2)}{e^{-\xi_1} + \delta_1 \cos(\xi_2) + e^{\xi_1}\delta_2},$$

where $\xi_1 = b_1 y + c_1 z, \xi_2 = a_2 x + (-a_2 - c_2)y + c_2 z + a_2^3 t.$

Case 3

$$a_1 = a_1, b_1 = -a_1 - c_1, c_1 = c_1, d_1 = -a_1^3,$$
$$a_2 = 0, b_2 = b_2, c_2 = c_2, d_2 = 0, \delta_1 = \delta_1, \delta_2 = \delta_2,$$

where $a_1, b_2, c_1, c_2, \delta_1, \delta_2$ are arbitrary real constants.

By the same method as above,

$$u_3(x,y,z,t) = -\frac{2(-a_1 e^{-\xi_1} + a_1\delta_2 e^{\xi_1})}{e^{-\xi_1} + \delta_1 \cos(\xi_2) + e^{\xi_1}\delta_2},$$

can be obtained, where $\xi_1 = a_1 x + (-a_1 - c_1)y + c_1 z - a_1^3 t, \xi_2 = b_2 y + c_2 z.$

Taking u_1 as an example. The superposition process of solitary waves and periodic wave is shown in Figure 1. We can see that the amplitude of the superposition is from strong to weak and then to strong, and the change of amplitude is affected by the frequency and wavelength of the propagation process. At the same time, the nature of the solitons not being deformed and not being destroyed when interacting can also be seen directly from the figures.

2.2. Interaction Solutions between Lump Wave and Solitary Waves

Assume that the form of the solutions of Eq. (3) is:

$$f(x,y,z,t) = g^2 + h^2 + l + \mu, \tag{5}$$

$$\begin{cases} g &= a_3 x + b_3 y + c_3 z + d_3 t \\ h &= a_4 x + b_4 y + c_4 z + d_4 t \\ l &= k \cdot \cosh(a_5 x + b_5 y + c_5 z + d_5 t) \end{cases}$$

where $a_i, b_i, c_i, d_i (i = 3,4,5)$, k and μ are undetermined constant. Substituting Eq. (5) into Eq. (3) and equating the corresponding coefficients of $\cosh(a_5 x + b_5 y + c_5 z + d_5 t), \sinh(a_5 x + b_5 y + c_5 z + d_5 t)$, $\cosh(a_5 x + b_5 y + c_5 z + d_5 t)^2, \sinh(a_5 x + b_5 y + c_5 z + d_5 t)^2$ and constant term to zero. Then we obtain the following cases.

Case 1

$$k = k, a_3 = a_3, b_3 = -\frac{a_3^2 + c_3 a_3 + a_4^2 + a_4 b_4 + a_4 c_4}{a_3}, c_3 = c_3, d_3 = 0, a_4 = a_4,$$
$$b_4 = b_4, c_4 = c_4, d_4 = 0, a_5 = a_5, b_5 = -a_5 - c_5, c_5 = c_5, d_5 = -a_5^3, \mu = \mu,$$

where $a_3, a_4, a_5, b_4, c_3, c_4, c_5, k$ and μ are arbitrary real constants.

We bring these values into Eq. (5) and get the solution of Eq. (1) with the following form:

$$u_4(x,y,z,t) = -\frac{2(2a_3 g + 2a_4 h + k a_5 \sinh(a_5 x + (-a_5 - c_5)y + c_5 z - a_5^3 t))}{g^2 + h^2 + l + \mu},$$

where

$$\begin{cases} g = a_3x - \dfrac{a_3^2 + c_3a_3 + a_4^2 + a_4b_4 + a_4c_4}{a_3}y + c_3z \\ h = a_4x + b_4y + c_4z \\ l = k\cdot\cosh(a_5x + (-a_5 - c_5)y + c_5z - a_5^3t). \end{cases}$$

Case 2

$$k = k, a_3 = 0, b_3 = b_3, c_3 = c_3, d_3 = 0, a_4 = 0,$$
$$b_4 = b_4, c_4 = c_4, d_4 = 0, a_5 = a_5, b_5 = -a_5 - c_5, c_5 = c_5, d_5 = -a_5^3, \mu = \mu,$$

where $a_5, b_3, b_4, c_3, c_4, c_5, k$ and μ are arbitrary real constants.

In the same way, we can obtain

$$u_5(x,y,z,t) = -\frac{2ka_5\sinh(a_5x + (-a_5 - c_5)y + c_5z - a_5^3t)}{g^2 + h^2 + l + \mu},$$

where

$$\begin{cases} g &= b_3y + c_3z \\ h &= b_4y + c_4z \\ l &= k\cdot\cosh(a_5x + (-a_5 - c_5)y + c_5z - a_5^3t). \end{cases}$$

Case 3

$$k = k, a_3 = 0, b_3 = b_3, c_3 = c_3, d_3 = 0, a_4 = -b_4 - c_4, b_4 = b_4,$$
$$c_4 = c_4, d_4 = 0, a_5 = a_5, b_5 = -a_5 - c_5, c_5 = c_5, d_5 = -a_5^3, \mu = \mu,$$

where $a_5, b_3, b_4, c_3, c_4, c_5, k$ and μ are arbitrary real constants.

By the same method as above, we have

$$u_6(x,y,z,t) = -\frac{2(2(-b_4 - c_4)h + ka_5\sinh(a_5x + (-a_5 - c_5)y + c_5z - a_5^3t))}{g^2 + h^2 + l + \mu},$$

where

$$\begin{cases} g &= b_3y + c_3z \\ h &= (-b_4 - c_4)x + b_4y + c_4z \\ l &= k\cdot\cosh(a_5x + (-a_5 - c_5)y + c_5z - a_5^3t). \end{cases}$$

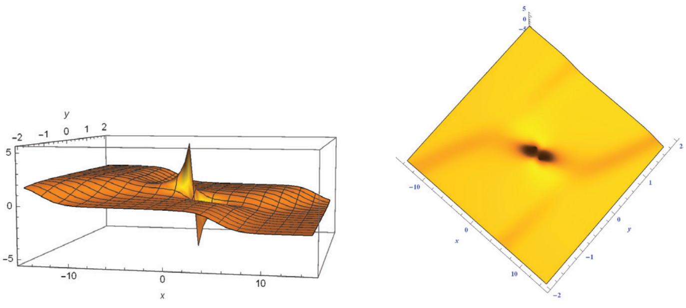
Observing Figures 2–4, we know that these images are describing the superposition process of lump waves and solitary waves. Taking u_4 as an example, we can see intuitively that the lump wave of Eq. (1) retains its shape, amplitude and some other physical properties after interacting with the solitons. In other words, the images show that the interaction is elastic.

3. Exact Meromorphic Solutions by Complex Method

3.1. Preliminary Lemmas

Let $m \in \mathbb{N} := \{1,2,3...\}, r = (r_0, r_1, ..., r_m), i = 0,1,...,m,$ and

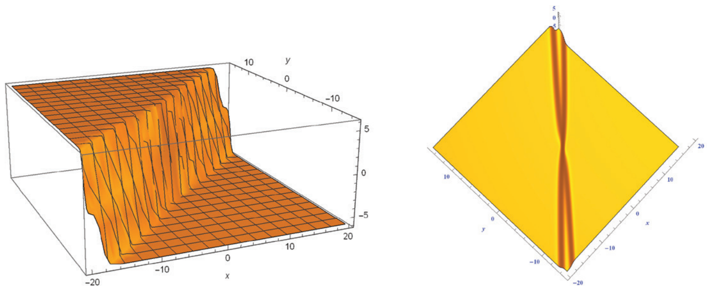
$$M_r[U](z) := \prod_{i=0}^{m}[U^{(i)}(z)]^{r_i}.$$



(a) 3D image (b) overhead view

(c) contour map

Figure 2: u_4 with $\mu = z = t = 0, a_3 = a_5 = b_4 = c_5 = 1, a_4 = 4, c_3 = 5, c_4 = 6, k = 2$



(a) 3D image (b) Overhead view

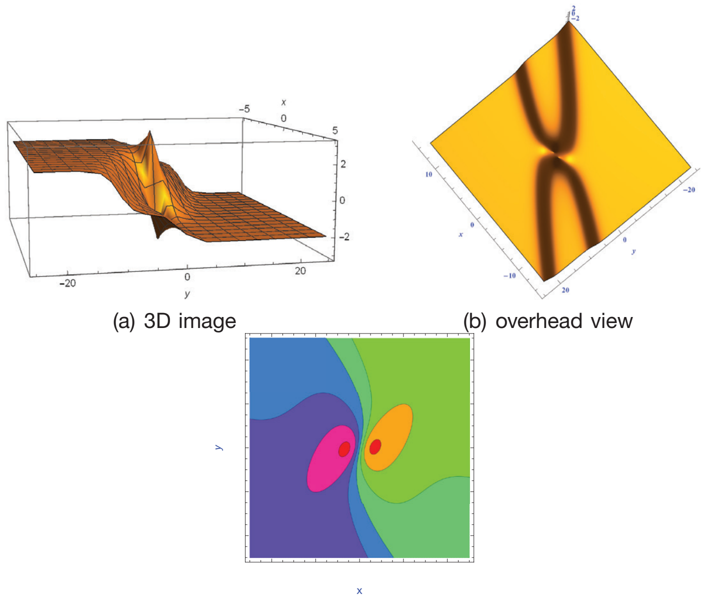
(c) Contour map

Figure 3: u_5 with $\mu = z = t = 0, b_3 = b_4 = c_3 = c_4 = c_5 = 1, a_5 = 3, k = 5$

$p(r) := r_0 + r_1 + \cdots + r_m$ is called the degree of $M_r[U]$. The differential polynomial is defined as:

$$P(U, U', \ldots, U^{(m)}) = \sum_{r \in I} a_r M_r[U],$$

where I is a finite index set, and a_r are constants. Then deg $P(U, U', \ldots, U^{(m)}) := max_{r \in I}\{p(r)\}$. Considering the following complex ordinary differential equation:



(a) 3D image (b) overhead view

(c) contour map

Figure 4: u_6 with $a_5 = -1, \mu = z = t = 0, b_3 = b_4 = c_4 = 1, c_3 = c_5 = k = 2$

$$P(U, U', \ldots, U^{(m)}) = bU^n + c, \tag{6}$$

where $b \neq 0$, c is constant, $n \in \mathbb{N}$

Definition 1. Let $p, q \in \mathbb{N}$ Suppose that the meromorphic solution $U(z)$ of Eq. (6) has at least one pole, we say that Eq. (6) satisfies the weak <*p*, *q*> condition if substituting Laurent series

$$U = \sum_{k=-q}^{\infty} c_k z^k, q > 0, c_{-q} \neq 0 \tag{7}$$

into Eq. (6) and then we can determine p distinct Laurent singular parts $\sum_{k=-q}^{-1} c_k z^k$.

Definition 2. Let ω_1, ω_2 be two given complex numbers, such that $Im \frac{\omega_1}{\omega_2} > 0, L = L[2\omega_1, 2\omega_2]$ be discrete subset $L[2\omega_1, 2\omega_2] = \{\omega \mid \omega = 2n\omega_1 + 2m\omega_2, n, m \in \mathbb{Z}\}$, which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ The discriminant $\Delta = \Delta(c_1, c_2) := c_1^3 - 27c_2^2$ and

$$s_n = s_n(L) := \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^n}.$$

The Weierstrass elliptic function $\wp(z) := \wp(z, g_2, g_3)$ is a meromorphic function with double periods $2\omega_1, 2\omega_2$ and satisfying:

$$[\wp'(z)]^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

where $g_2 = 60s_4, g_3 = 140s_6$, and $\Delta(g_2, g_3) \neq 0$.

Definition 3. The Weierstrass zeta function $\zeta(z)$ is also a meromorphic function which satisfies

$$\wp(z) = -\zeta'(z),$$
$$\zeta(z - z_0) = \zeta(z) - \zeta(z_0) + \frac{1}{2}\left[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)}\right].$$

Definition 4. A meromorphic function f belongs to the class W [20], [21] if f is a rational function of z, or a rational function of $e^{\alpha z}, \alpha \in \mathbb{C}$, or an elliptic function.

The k order Briot-Bouquet equation (BBEq) is defined as

$$F(U^{(k)}, U) = \sum_{i=0}^{n} P_i(U)(U^{(k)})^i = 0,$$

where $k \in \mathbb{N}, P_i(U)$ is a polynomial with constant coefficients.

This equation has been studied by Eremenko and his collaborators [21], and their results are widely recognized by scholars.

Lemma 1. *[22], [23], [24], [25], [26] Let $p, I, m, n \in \mathbb{N}, degF(U^{(m)}, U) < n$. Suppose that an m order BBEq Eq. (6) satisfies weak <p, q> comdition, then the meromorphic solution U belongs to class W. If for some values of parameters such solution U exists, then other meromorphic solutions form a one-parametric family $U(z - z_0), z_0 \in \mathbb{C}$. Furthermore, each rational function solution $U = R(z)$ is of the form*

$$R(z) = \sum_{i=1}^{I}\sum_{j=1}^{q} \frac{c_{ij}}{(z - z_i)^j} + c_0,$$

with $I(\le p)$ distinct poles of multiplicity q

Each simply periodic solution is a rational function $R(\eta)$ of $\eta = e^{\alpha z}$ $(\alpha \in \mathbb{C})$. $R(\eta)$ has $I(\le p)$ distinct poles of multiplicity q and is of the form

$$R(\eta) = \sum_{i=1}^{I}\sum_{j=1}^{q} \frac{c_{ij}}{(\eta - \eta_i)^j} + c_0.$$

Each elliptic solution with pole at $z = 0$ can be written as

$$\begin{aligned} U(z) \quad &= \sum_{i=1}^{I-1}\sum_{j=2}^{q} \frac{(-1)^j c_{-ij}}{(j-1)!} \frac{d^{j-2}}{dz^{j-2}} \left(\frac{1}{4}\left[\frac{\wp'(z) + B_i}{\wp(z) - A_i} \right]^2 - \wp(z) \right) \\ &+ \sum_{i=1}^{I-1} \frac{c_{-ij}}{2} \frac{\wp'(z) + B_i}{\wp(z) - A_i} + \sum_{j=2}^{q} \frac{(-1)^j c_{-ij}}{(j-1)!} \frac{d^{j-2}}{dz^{j-2}} \wp(z) + c_0, \end{aligned}$$

where c_{-ij} are given by Eq. (7), $B_i^2 = 4A_i^3 - g_2 A_i - g_3$, and $\sum_{i=1}^{I} c_{-i1} = 0$.

3.2. Apply Lemmas to Solve Solutions

Taking traveling wave transformation

$$u(x, y, z, t) = U(s), s = kx + ly + mz + nt,$$

to Eq. (1) and conducting some simple calculations, we have

$$k^3 U^{(3)} + nU' + k^2 (U')^2 + \lambda = 0. \tag{8}$$

On this basis, let $V = U'$, we can get

$$k^3 V'' + nV + k^2 V^2 + \lambda = 0. \tag{9}$$

Considering Eq. (9) in the complex plane, its solutions are included in class W. It is found by calculation that the Eq. (9) satisfies the weak <1, 2> condition.

It's a guide for assuming the form of the solutions. Furthermore, by using the complex method, the corresponding new exact solutions of Eq. (1) will be obtained.

So suppose $k+l+m \neq 0$ and $k \neq 0$, the solutions of Eq. (9) can be divided into the following three cases.

Case 1. Suppose the rational function solutions of Eq. (9) with pole at s = 0 are

$$V_r(s) = \frac{A}{s^2} + \frac{B}{s} + C. \tag{10}$$

Substituting Eq. (10) into Eq. (9), we have $A = -6k, B = 0, C = -\frac{n}{2k^2}, \lambda = \frac{n^2}{4k^2}$.

So the rational function solutions of Eq. (9) are:

$$V_r(s) = -\frac{6k}{(s-s_0)^2} - \frac{n}{2k^2}, s_0 \in \mathbb{C},$$

and all rational solutions of Eq. (8) can be expressed as

$$U_r(s) = \frac{6k}{s-s_0} - \frac{n}{2k^2}(s-s_0) + c_1,$$

where $s_0 \in \mathbb{C}$, c_1 is constant, $\lambda = \frac{n^2}{4k^2}$.

Case 2. Let $\eta = e^{\alpha s}$. Substituting $R(\eta)$ into Eq. (9) yields

$$k^3\alpha^2(R''\eta^2 + R'\eta) + nR + k^2R^2 + \lambda = 0. \tag{11}$$

We assume the simply periodic solutions of Eq. (11) with pole at $s = 0$ are

$$R(\eta) = \frac{X}{(\eta-1)^2} + \frac{Y}{\eta-1} + Z.$$

By the method of undetermined coefficients, we obtain $X = -6k\alpha^2, Y = -6k\alpha^2, Z = \frac{-n-k^3\alpha^2}{2k^2}, \lambda = \frac{n^2-k^6\alpha^4}{4k^2}$.

So all simply periodic solutions of Eq. (9) are

$$V_s(s) = \frac{-6k\alpha^2}{(e^{\alpha(s-s_0)}-1)^2} + \frac{-6k\alpha^2}{e^{\alpha(s-s_0)}-1} - \frac{n+k^3\alpha^2}{2k^2},$$

where $s_0 \in \mathbb{C}, \alpha$ is constant. Integrating $V_s(s)$ once and getting the solutions of Eq. (8):

$$U_s(s) = \frac{6k\alpha}{e^{\alpha(s-s_0)}-1} - \frac{n+k^3\alpha^2}{2k^2}(s-s_0) + c_2,$$

where $s_0 \in \mathbb{C}, \alpha$ and c_2 are constants, $\lambda = \frac{n^2-k^6\alpha^4}{4k^2}$.

Case 3. We can directly assume that the elliptic solutions of Eq. (9) with pole at s = 0 are

$$V_d(s) = c_{-2}\wp(s) + c_0.$$

Together with Eq. (9), we get $c_{-2} = -6k, c_0 = -\frac{n}{2k^2}$.

Therefore, all elliptic solutions of Eq. (9) are in form of

$$V_d(s) = -6k\wp(s-s_0) - \frac{n}{2k^2}, s_0 \in \mathbb{C},$$

and the solutions of Eq. (8) are:

$$U_d(s) = 6k\zeta(s) + 3k\left(\frac{\wp'(s)+B}{\wp(s)-A}\right) - \frac{n}{2k^2}(s-s_0) - 6k\zeta(s_0) + c_3,$$

where $s_0 \in \mathbb{C}$, c_3 is constant, $B^2 = 4A^3 - g_2A - g_3$, A and g_2 are arbitrary constants, $\lambda = -3k^4 g_2 + \frac{n^2}{4k^2}$.

The properties of $U_r(s)$ and $U_s(s)$ are shown in Figures 5 and 6. Looking at Figures 5 and 6, we find that there is a crack in all the images. This crack is actually a line of countless poles, and it also moves forward with time. The values on both sides of the line tend to ∞.

4. Conclusions

In this paper, a number of periodic solitary wave solutions, the interaction solutions between lump wave and solitary waves and the meromorphic solutions of the new (3+1)-dimensional BLMP equation are obtained by means of symbolic calculation of mathematical software and complex method. The soliton solutions are brought back to the original equation to calculate and confirm that they are correct. Since $q = 2, c_{-2} = -6k$ can be obtained by bringing the Laurent series into Eq. (7) and combining the highest order balance method, Eq. (7) is strictly in accordance with the $<p, q>$ condition. This is also the guarantee for the accuracy of mathematical models such as rational function solutions, simply periodic solutions and elliptic solutions. And the consideration of solutions in the complex plane enriches the research of NLPDEs and also expands the types of solutions. In addition, we should know that the real-valued solutions in class W can be obtained on the basis of the exact meromorphic solutions [22]. The real-valued solutions are of great significance both in complex analysis and in the field of physics. These solutions have not been mentioned by other scholars before and their properties are clearly shown in Figures 1–6. We also describe the meaning of the images in detail in the text. The methods mentioned in this paper are also applicable to other evolution equations. In

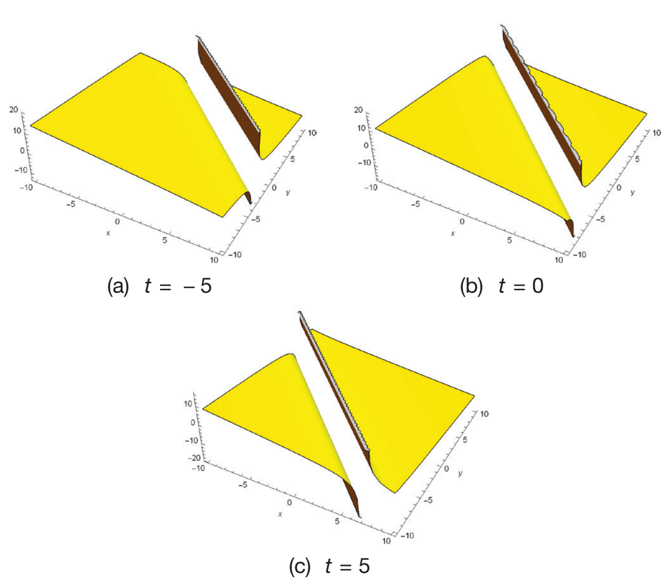


Figure 5: $U_r(s)$ with $m = -1, k = l = n = 1, z = 2, s_0 = 0$

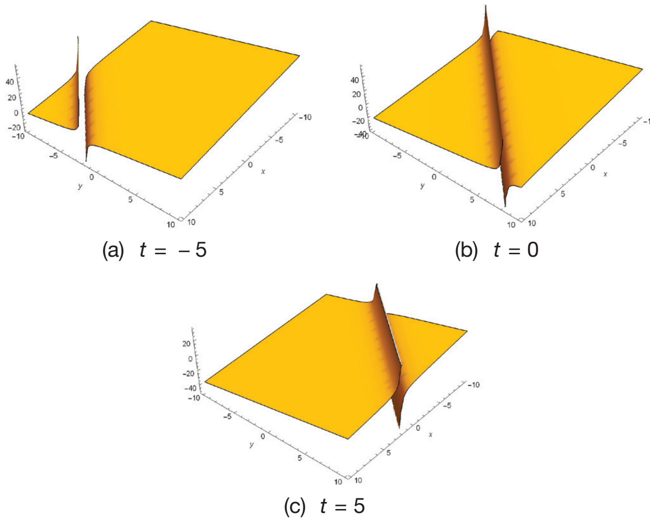


Figure 6: $U_s(s)$ with $n=-2, k=-1, l=m=\alpha=1, z=2, s_0=0$

future research, we can also use some promising numerical methods, such as Keller Box [27], [28], Shooting method [29] and neural networks [30], [31], to obtain the numerical solution of the equation. The numerical solution obtained by using the approximate method can be used to help solve practical problems in life.

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