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Some new existence results on the hybrid fractional differential equation with variable order derivative

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Abstract

Fractional order systems play a vital role in the study of the abnormal behavior of dynamic systems in physics, biology, viscoelasticity, and in the study of population dynamics. The thing that caught our attention thinking about using the order of the fractional derivatives as a function, where we find some works (mentioned in the introduction), in which the order of the fractional derivative has been used as a function that changes with concentration, time, space, or other independent quantities. Motivated by these works in this manuscript, we studied the existence of solutions of the following hybrid fractional differential equation of variable order involving the ψ-Hilfer fractional derivative

$$\begin{cases} {}^{H}D_{0_+}^{\alpha(t),\sigma,\psi}\left(\dfrac{u(t)}{f(t,{}^{H}D_{0_+}^{\beta(t),\sigma,\psi}u(t))}\right) = h(t,{}^{H}D_{0_+}^{\beta(t),\sigma,\psi}u(t)), \quad t\in[0,T] \\ (\psi(t)-\psi(0))^{1-\zeta(t)}u(t)\,|_{t=0} = u_0, \quad u_0\in\mathbb{R} \end{cases}$$

where $0<\beta(t)<\alpha(t)<1$, $0<\sigma<1$, $\zeta(t)=\alpha(t)+\sigma(1-\alpha(t))$, $f\in C(J\times\mathbb{R},\mathbb{R}^*)$, and $h\in C(J\times\mathbb{R},\mathbb{R})$.

We used some techniques to show the existence of the solution by the Krasnoselskii fixed point theorem. These techniques are based on the change of the variable ${}^{H}D_{0_+}^{\beta(t),\sigma,\psi}u(t)$. Further, an example is provided to illustrate our results.

Keywords: ψ-Hilfer derivative; variable order fractional derivative; Krasnoselskii fixed point theorem.

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1. Introduction

The theory of fractional differential equations has emerged as an interesting area to explore in recent years. The concepts of fractional derivation and integration are often associated with Riemann and Liouville, while the questioning of the generalization of the notion of derivative to fractional orders are examined in earlier studies. Indeed, the history of fractional calculus began with a key question from Leibniz, to whom we owe the idea of fractional derivation. He introduces the derivation symbol of order n, $\frac{d^n y}{x^n}$, where n is a positive integer. It was perhaps a naive game of symbols that prompted the L'H'ôpital. To wonder about the possibility of having n in $\mathbb{Q}$, He asked the question: what if $n = \frac{1}{2}$ In 1695, in a letter to the L'H'ôpital, Leibniz prophetically wrote: "So it follows that $D^{\frac{1}{2}}x$ will be equal to $x\sqrt[2]{dx : x}$, an apparent paradox from which useful conclusions will one day be drawn?" Fractional calculus has been considered one of the best mathematical tools to characterize the memory property of complex systems and certain materials [16, 2, 6]. Indeed, in the classical approach, the memory of a system can be represented by the following integer order derivative

$$\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t) - f(t - \Delta t)}{\Delta t}, \quad 0 < t,$$

when we use this definition to characterize a system, we can say that it represents the short memory property of the system, since this definition only used values in two points. However, in the fractional approach, the memory of a system can be represented by the following fractional order derivative [7]

$$\frac{d^\alpha f(t)}{dt^\alpha} = \lim_{\tau \to 0} \tau^{-\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f(t - k\tau), \quad n\tau = t, 0 < \alpha < 1,$$

where α is the fractional derivative order. the expression of fractional order derivative exhibits the memory of the considered function's history because when calculating the value of the fractional derivative at t, all the previous values of the function $f(t)$ must be used. For more details see [14]. The fractional calculus allowed the operations of integration and differentiation to all the complex orders. This fact allows us to consider the order of fractional derivatives as a function of time, space, or other variables, rather than a constant of arbitrary order [5], [10]. Samko et al. first proposed the concept of variable order operator and investigated the mathematical properties of variable order integration and differentiation operators of Riemann Liouville type [4]. there are principally two types of variable-order fractional derivative definitions, with the details about the two definitions given in [14]. Lorenzo and Hartley made some theoretical studies via the iterative Laplace transform and generalized different types of variable order fractional operator definitions [5].

In [15] the author studied the existence and uniqueness of a solution to an initial value problem for the following differential equation of variable order involving the Riemann Liouville derivative

$$\begin{cases} D_{0^+}^{\alpha(t)} x(t) = f(t, x(t)), \quad 0 < t \le T \\ x(0) = 0 \end{cases} \tag{1}$$

where

$$D_{0^+}^{\alpha(t)} x(t) = \frac{d}{dt} \int_0^t \frac{(t-s)^{-\alpha(t)}}{\Gamma(1 - \alpha(t))} x(s) ds, \quad t > 0$$

The authors in [11] studied the existence and Ulam-stability of solution of the following fractional order hybrid differential equations to the one with variable order involving the Caputo derivative

$$\begin{cases} D_0^{\alpha(t)} (x(t) - f(t, x(t))) = g(t, x(t)), \quad t \in [0, T] \\ x(0) = x_0 \end{cases} \tag{2}$$

where

$$D_0^{\alpha(t)}x(t)=\int_0^t \frac{(t-s)^{-\alpha(t)}}{\Gamma(1-\alpha(t))}x'(s)ds.$$

Perturbation techniques are compelling in the study of nonlinear systems which are difficult to be studied and solved; however, the perturbation of such a problem allows us to study this problem and its characteristics. These techniques differ depending of the problem studied, where we find linear perturbation, quadratic perturbation, and implicit perturbation. For further details on these types of perturbations, see [17]. In this paper, we are interested in the study of a quadratic-ally perturbed problem, problems perturbed in this way are called hybrid differential equations. For more details on these types of these equations see [18], [17], [19]. We investigate the existence of solutions, on a sub-interval of $J=[0,T]$, $(T<\infty)$ for the following hybrid fractional differential equation involving ψ-Hilfer fractional derivative

$$\begin{cases} {}^H D_{0_+}^{\alpha(t),\sigma,\psi}\left(\dfrac{u(t)}{f(t,{}^H D_{0_+}^{\beta(t),\sigma,\psi}u(t))}\right)=h(t,{}^H D_{0_+}^{\beta(t),\sigma,\psi}u(t)), \quad t\in[0,T] \\ (\psi(t)-\psi(0))^{1-\zeta(t)}u(t)\,|_{t=0}=u_0 \quad u_0\in\mathbb{R} \end{cases} \tag{3}$$

where $0<\beta(t)<\alpha(t)<1$, $0<\sigma<1$, $\zeta(t)=\alpha(t)+\sigma(1-\alpha(t))$, ${}^H D_{0_+}^{\alpha(t),\sigma,\psi}(.)$, ${}^H D_{0_+}^{\beta(t),\sigma,\psi}(.)$ denote ψ-Hilfer fractional derivatives of variable order $\alpha(t)$, $\beta(t)$ and type σ, $f\in C(J\times\mathbb{R},\mathbb{R}^*)$, and $h\in C(J\times\mathbb{R},\mathbb{R})$.

2. Preliminaries

Let $J=[0,T]$ be a finite interval of $\mathbb{R}$. $C(J,\mathbb{R})$ be the Banach space of continuous real function h with the norm $\| h \|=max\{|h(t)|:t\in J\}$. $C^n(J,\mathbb{R})$ be the Banach space of n-times continuously differentiable functions on J.

Let $[a,b]$ with $(0<a<b<\infty)$ be a finite interval and $\psi\in C^1([a,b])$ be increasing function such that $\psi'(t)\neq 0$, $\forall t\in[a,b]$, we consider the weighted space:

$$C_{1-\zeta,\psi}([a,b])=\left\{\Phi(t):(a,b]\to\mathbb{R},(\psi(t)-\psi(a))^{1-\zeta}\Phi(t)\in C^1[a,b]\right\}$$

endowed with the norm:

$$\| \Phi(t) \|_{C_{1-\zeta,\psi}([a,b])}=max_{t\in[a,b]}\,|(\psi(t)-\psi(a))^{1-\zeta}\Phi(t)|$$

Let us recall some definitions and properties of fractional calculus.

Definition 2.1. *[1] The left-sided ψ-Riemann-Liouville fractional integral of variable order $\alpha:[a,b]\to(0,+\infty)$, for an integrable function $\Phi:[a,b]\to\mathbb{R}$ with respect to another function $\psi:[a,b]\to\mathbb{R}$, which is an increasing differentiable function such that $\psi'(t)\neq 0$, for all $t\in[a,b]$, $(-\infty<a<b\leq+\infty)$, is defined as the following two types:*

$$I_{a^+}^{\alpha(t);\psi}\Phi(t)=\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha(t)-1}}{\Gamma(\alpha(t))}\Phi(s)ds \tag{4}$$

$$I_{a^+}^{\alpha(t);\psi}\Phi(t)=\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha(s)-1}}{\Gamma(\alpha(s))}\Phi(s)ds \tag{5}$$

where $\Gamma(.)$ is the Euler gamma function defined by

$$\Gamma(z)=\int_0^{+\infty} e^{-t}t^{z-1}dt, z>0$$

Definition 2.2. *[1] The left-sided ψ-Riemann-Liouville fractional derivative of variable order $\alpha:[a,b]\to(n-1,n]$ with $n=[\alpha]+1$, for an integrable function $\Phi:[a,b]\to\mathbb{R}$ with respect to another function $\psi:[a,b]\to\mathbb{R}$, which is an increasing differentiable function such that $\psi'(t)\neq 0$, for all $t\in[a,b]$, $(-\infty<a<b\leq+\infty)$, is defined as the following two types:*

$$D_{a^+}^{\alpha(t);\psi}\Phi(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha(t);\psi}\Phi(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\int_a^t\Phi\,\frac{\psi'(s)(\psi(t)-\psi(s))^{n-\alpha(t)-1}}{\Gamma(n-\alpha(t))}\Phi(s)ds \tag{6}$$

$$D_{a^+}^{\alpha(t);\psi}\Phi(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha(t);\psi}\Phi(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\int_a^t\Phi\,\frac{\psi'(s)(\psi(t)-\psi(s))^{n-\alpha(s)-1}}{\Gamma(n-\alpha(s))}\Phi(s)ds \tag{7}$$

Definition 2.3. *[13] The left-sided ψ-Caputo fractional derivative of variable order $\alpha:[a,b]\to(n-1,n]$ with $n=[\alpha]+1$, for a function $\Phi\in C^n([a,b])$ with respect to another function $\psi:[a,b]\to\mathbb{R}$, which is an increasing differentiable function such that $\psi'(t)\neq 0$, for all $t\in[a,b]$, $(-\infty<a<b\leq+\infty)$, is defined as the following two types:*

$${}^{C}D_{a^+}^{\alpha(t);\psi}\Phi(t)=I_{a_+}^{n-\alpha(t);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\Phi(t)=\int_a^t\Phi\,\frac{\psi'(s)(\psi(t)-\psi(s))^{n-\alpha(t)-1}}{\Gamma(n-\alpha(t))}\Phi_\psi^{[n]}(s)ds \tag{8}$$

$${}^{C}D_{a^+}^{\alpha(t);\psi}\Phi(t)=I_{a_+}^{n-\alpha(t);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\Phi(t)=\int_a^t\Phi\,\frac{\psi'(s)(\psi(t)-\psi(s))^{n-\alpha(s)-1}}{\Gamma(n-\alpha(s))}\Phi_\psi^{[n]}(s)ds \tag{9}$$

where $\Phi_\psi^{[n]}(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\Phi(t)$.

Definition 2.4. *[8] Let $\alpha:[a,b]\to(n-1,n]$, with $n=[\alpha]+1,\psi\in C^n([a,b],\mathbb{R})$ a function such that $\psi(t)$ is increasing and $\psi'(t)\neq 0$ for all $t\in[a,b]$.*

The left-sided ψ-Hilfer fractional derivative of function $\Phi\in C^n([a,b],\mathbb{R})$ of order $\alpha(t)$ and type $\sigma\in[0,1]$ is determined as

$${}^{H}D_{a_+}^{\alpha(t),\sigma,\psi}\Phi(t)=I_{a_+}^{\sigma(n-\alpha(t));\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a_+}^{(1-\sigma)(n-\alpha(t));\psi}\Phi(t)$$

In other way

$${}^{H}D_{a_+}^{\alpha(t),\sigma,\psi}\Phi(t)=I_{a_+}^{\sigma(n-\alpha(t));\psi}D_{a_+}^{\zeta(t),\psi}\Phi(t)$$

where

$$D_{a_+}^{\zeta,\psi}\Phi(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a_+}^{(1-\sigma)(n-\alpha(t));\psi}\Phi(t)$$

with $\zeta(t)=\alpha(t)+\sigma(n-\alpha(t))$

In particular, the ψ-Hilfer fractional derivative of variable order $\alpha(t)\in(0,1)$ and type $\sigma\in[0,1]$, can be written in the following two types:

$${}^{H}D_{a^+}^{\alpha(t);\sigma;\psi}\Phi(t)=\int_a^t\frac{\psi'(s)(\psi(t)-\psi(s))^{\zeta(t)-\alpha(t)-1}}{\Gamma(\zeta(t)-\alpha(t))}D_{a^+}^{\zeta(s),\psi}\Phi(s)ds \tag{10}$$

$${}^{H}D_{a^+}^{\alpha(t);\sigma;\psi}\Phi(t)=\int_a^t\Phi\,\frac{\psi'(s)(\psi(t)-\psi(s))^{\zeta(s)-\alpha(s)-1}}{\Gamma(\zeta(s)-\alpha(s))}D_{a^+}^{\zeta(s),\psi}\Phi(s)ds \tag{11}$$

where $\zeta(t) = \alpha(t) + \sigma(1-\alpha(t))$, and $D_{a^+}^{\zeta(t),\psi}\Phi(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{1-\zeta(t);\psi}\Phi(t)$.

We notice that, if the order $\alpha(t)$ is a constant function p, we know there are some important properties as following. Let $J = [0,T]$, $(T < +\infty)$.

Lemma 2.1. *[1]. Let* $p,q > 0$. *Then we have*

i) $I_{0_+}^{p;\psi}(\psi(t)-\psi(0))^{q-1} = \frac{\Gamma(q)}{\Gamma(q+p)}(\psi(t)-\psi(0))^{p+q-1}$

ii) ${}^H D_{0_+}^{p;\sigma;\psi}(\psi(t)-\psi(0))^{\zeta-1} = 0$

Lemma 2.2. *[12]. Let* $p > 0$ *and* $q > 0$. *Then the relation*

$$I_{0_+}^{p;\psi} I_{0_+}^{q;\psi} h(t) = I_{0_+}^{p+q;\psi} h(t)$$

holds almost everywhere for $t \in J$ and $h \in L^m(J,\mathbb{R})$, $m \geq 1$.

Lemma 2.3. *[8] Let* $h \in C^n[0,T]$, $n-1 < p < n$, $0 \leq \sigma \leq 1$, *and* $\zeta = p + \sigma(n-p)$. *Then for all* $t \in J$

$${}^H D_{0_+}^{p;\sigma;\psi} I_{0_+}^{p;\psi} h(t) = h(t)$$

and

$$I_{0_+}^{p;\psi\,H} D_{0_+}^{p;\sigma;\psi} h(t) = h(t) - \sum_{k=1}^{n} \frac{(\psi(t)-\psi(a))^{(\zeta-k)}}{\Gamma(\varsigma - k + 1)} h_\psi^{(n-k)} I_{0_+}^{(1-\sigma)(n-p);\psi} h(0)$$

where $h_\psi^{(n-k)} h(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n-k} h(t)$ in particular, if $0 < p < 1$, we have

$$I_{0_+}^{p;\psi\,H} D_{0_+}^{p;\sigma;\psi} h(t) = h(t) - \frac{(\psi(t)-\psi(0))^{\zeta-1}}{\Gamma(\varsigma)} I_{0_+}^{(1-\sigma)(1-p);\psi} h(0).$$

Moreover, if $h \in C_{1-\zeta}(J)$ and $I_{0_+}^{1-\zeta;\psi} h \in C_{1-\zeta,\psi}^1(J)$ such that $0 < \zeta < 1$. Then for all $t \in J$

$$I_{0_+}^{\zeta;\psi\,H} D_{0_+}^{\zeta;\sigma;\psi} h(t) = h(t) - \frac{(\psi(t)-\psi(0))^{\zeta-1}}{\Gamma(\zeta)} I_{0_+}^{(1-\zeta);\psi} h(0)$$

Lemma (2.2) is a semi-group property for the Riemann-Liouville fractional integral, which is very crucial in obtaining the Lemma (2.3) which allows us to determine the existence of the solutions of differential equations for the ψ-Hilfer fractional derivative. For general functions $\alpha(t)$, $\rho(t)$, we notice that the semi-group property does not hold, i.e., $I_{0_+}^{\alpha(t);\psi} I_{0_+}^{\rho(t);\psi} h(t) \neq I_{0_+}^{\alpha(t)+\rho(t);\psi} h(t)$. Then we have the following Lemma

Lemma 2.4. *Let h(t),* $\alpha(t)$, $\beta(t)$ *be real functions on a finite interval* $[0,T]$ *and* $\psi : [0,T] \to \mathbb{R}$, *that is an increasing differentiable function such that* $\psi'(t) \neq 0$, *for all* $t \in [0,T]$, *then we have*

$$I_{0_+}^{\alpha(t);\psi} I_{0_+}^{\beta(t);\psi} h(t) \neq I_{0_+}^{\alpha(t)+\beta(t);\psi} h(t), \quad t \in [0,T]$$

in particular, for general functions $0 < \alpha(t) < 1$, we have

$$I_{0_+}^{\alpha(t);\psi} I_{0_+}^{1-\alpha(t);\psi} h(t) \neq I_{0_+}^{\alpha(t)+1-\alpha(t);\psi} h(t) = I_{0_+}^{1;\psi} h(t), \quad t \in [0,T]$$

Example

In this example, we will verify that $I_{0_+}^{\alpha(t);\psi} I_{0_+}^{\beta(t);\psi} h(t)\,|_{t=3} \neq I_{0_+}^{\alpha(t)+\beta(t);\psi} h(t)\,|_{t=3}$ for

$$\alpha(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 2 & 1 < t \le 3 \end{cases}$$

$$\beta(t) = \begin{cases} 2 & 0 \le t \le 1 \\ 3 & 1 < t \le 3, \end{cases}$$

$h(t) = 1$, and $\psi(t) = t$, then for $1 < t \le 3$ we have

$$\begin{aligned} I_{0_+}^{\alpha(t);\psi} I_{0_+}^{\beta(t);\psi} h(t) &= I_{0_+}^{\alpha(t)} I_{0_+}^{\beta(t)} h(t) \\ &= \int_0^1 \frac{(t-s)^{\alpha(t)-1}}{\Gamma(\alpha(t))} \int_0^s \frac{(s-\tau)^{2-1}}{\Gamma(2)} d\tau ds + \int_1^t \frac{(t-s)^{\alpha(t)-1}}{\Gamma(\alpha(t))} \int_0^s \frac{(s-\tau)^{3-1}}{\Gamma(3)} d\tau ds \\ &= \int_0^1 \frac{(t-s)^{\alpha(t)-1}}{2\Gamma(\alpha(t))} s^2 ds + \int_1^t \frac{(t-s)^{\alpha(t)-1}}{6\Gamma(\alpha(t))} s^3 ds \end{aligned}$$

then, for $t = 3$ we have

$$\begin{aligned} I_{0_+}^{\alpha(t)} I_{0_+}^{\beta(t)} h(t)\,|_{t=3} &= \int_0^1 \frac{(3-s)^{2-1}}{2\Gamma(2)} s^2 ds + \int_1^3 \frac{(3-s)^{2-1}}{6\Gamma(2)} s^3 ds \\ &= \frac{288}{120} \\ &= 2,4 \end{aligned}$$

on the other hande, we have

$$\begin{aligned} I_{0_+}^{\alpha(t)+\beta(t);\psi} h(t)\,|_{t=3} &= I_{0_+}^{\alpha(t)+\beta(t)} h(t)\,|_{t=3} \\ &= \int_0^3 \frac{(3-s)^{\alpha(3)+\beta(3)-1}}{\Gamma(\alpha(3)+\beta(3))} s^2 ds \\ &= \int_0^3 \frac{(3-s)^{2+3-1}}{\Gamma(2+3)} s^2 ds \\ &= \frac{2187}{12600} \simeq 0,173 \end{aligned}$$

therefore, we get

$$I_{0_+}^{\alpha(t);\psi} I_{0_+}^{\beta(t);\psi} h(t)\,|_{t=3} \neq I_{0_+}^{\alpha(t)+\beta(t);\psi} h(t)\,|_{t=3}$$

Lemma 2.5. *Let X be a Banach space. A mapping $A : X \to X$ is called a nonlinear contraction if there exists a continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\| Ax - Ay \| \le \phi(\| x - y \|)$, for all $x, y \in X$, which $\phi(r) < r$ for $r > 0$. In particular if $\phi(r) = cr$, $0 \le c \le 1$, then A is called a contraction on X with contraction constant c.*

Theorem 2.1. *[3] Let S be a closed, convex, and nonempty subset of the Banach algebra X. Suppose that $A, B : S \to X$ are two operators such that:*

- $Au + Bv \in S$ for all $u, v \in S$.
- A is a contraction on S
- B completely continuous on S.

Then, the operator equation $u = Au + Bu$ has a solution in S.

The following results are necessary for the analysis of our main result.

Let $\alpha:[0,T]\to(0,1)$, $\beta:[0,T]\to(0,1)$, and $P=\{[0,T_1],[T_1,T_2],[T_2,T_3],...,[T_{N-1},T]\}$. Thus, we define the functions $\alpha(t)$ and $\beta(t)$ as follows

$$\alpha(t)=\sum_{k=1}^{N}\alpha_k I_k(t), t\in[0,T] \tag{12}$$

$$\beta(t)=\sum_{k=1}^{N}\beta_k I_k(t), t\in[0,T] \tag{13}$$

where $0<\beta_k<\alpha_k<1$ and I_k is the indicator of the interval $[T_{k-1},T_k]$ ($k\in 1,2,...,N$, $T_0=0$ and $T_N=T$), that is $I_k(t)=1$ for $t\in[T_{k-1},T_k]$, $I_k(t)=0$ for elsewhere.

Then the functions $\alpha(t)$ and $\beta(t)$ can be written as follows

$$\alpha(t)=\begin{cases}\alpha_1, & t\in[0,T_1]\\ \alpha_2, & t\in[T_1,T_2]\\ . & \\ . & \\ . & \\ \alpha_N, & t\in[T_{N-1},T]\end{cases}$$

$$\beta(t)=\begin{cases}\beta_1, & t\in[0,T_1]\\ \beta_2, & t\in[T_1,T_2]\\ . & \\ . & \\ . & \\ \beta_N, & t\in[T_{N-1},T]\end{cases}$$

3. Existence Results

In this partition, we prove the existence of the solution to the given problem (3). We first present the following important result through which we can prove our major results.

According to (12) and (13), the problem (3) can be written by

$$\begin{cases}\sum_{k=1}^{N}I_k(t)\int_0^t\frac{\psi'(s)(\psi(t)-\psi(s))^{\zeta_k-\alpha_k-1}}{\Gamma(\zeta_k-\alpha_k)}D_{0_+}^{\zeta_k,\psi}\left(\frac{u(s)}{f(s,\chi_{\beta_k,u}(s))}\right)ds=h((t,\chi_{\beta_k,u}(t)), & t\in[0,T]\\ \sum_{k=1}^{N}I_k(t)(\psi(t)-\psi(0))^{1-\zeta_k}u(t)\,|_{t=0}=u_0, & u_0\in\mathbb{R}\end{cases} \tag{14}$$

where $\zeta_k=\alpha_k+\sigma(1-\alpha_k)$, and $D_{0_+}^{\zeta_k,\psi}\Phi(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{0_+}^{1-\zeta_k;\psi}\Phi(t)$

and $\chi_{\beta_k,u}(t)=\sum_{k=1}^{N}I_k(t)\int_0^t\frac{\psi'(s)(\psi(t)-\psi(s))^{\theta_k-\beta_k-1}}{\Gamma(\theta_k-\beta_k)}D_{0_+}^{\theta_k,\psi}u(s)ds,\quad \theta_k=\beta_k+\sigma(1-\beta_k)$, for all $t\in[0,T]$.

Hence, the equation

$$\sum_{k=1}^{N} I_k(t)\int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\zeta_k-\alpha_k-1}}{\Gamma(\zeta_k-\alpha_k)} D_{0_+}^{\zeta_k,\psi}\left(\frac{u(s)}{f(s,\chi_{\beta_k,u}(s))}\right)ds = h(t,\chi_{\beta_k,u}(t)), \quad t\in[0,T] \tag{15}$$

in the interval $[0,T_1]$ can be written by

$$\int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\zeta_1-\alpha_1-1}}{\Gamma(\zeta_1-\alpha_1)} D_{0_+}^{\zeta_1,\psi}\left(\frac{u(s)}{f(s,\chi_{\beta_1,u}(s))}\right)ds = h(t,\chi_{\beta_1,u}(t)), \quad t\in[0,T_1]. \tag{16}$$

Again the Equation (15) in the interval $[T_1,T_2]$ can be written by

$$\int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\zeta_2-\alpha_2-1}}{\Gamma(\zeta_2-\alpha_2)} D_{0_+}^{\zeta_2,\psi}\left(\frac{u(s)}{f(s,\chi_{\beta_2,u}(s))}\right)ds = h(t,\chi_{\beta_2,u}(t)), \quad t\in[T_1,T_2]. \tag{17}$$

In the same way in the interval $[T_{i-1},T_i]$, $i=3,4,...,N$ $(T_N=T)$ the equation (15) can be written by

$$\int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\zeta_i-\alpha_i-1}}{\Gamma(\zeta_i-\alpha_i)} D_{0_+}^{\zeta_i,\psi}\left(\frac{u(s)}{f(s,\chi_{\beta_i,u}(s))}\right)ds = h(t,\chi_{\beta_i,u}(t)), \quad t\in[T_{i-1},T_i] \tag{18}$$

Now, we are ready to state the definition of a solution to the problem (3), which is crucial in our work.

Definition 3.1. *We say that the problem* (3) *has a solution, if there exists* $u_1\in C_{1-\zeta_1,\psi}([0,T_1])$ *satisfying* (16) *and* $(\psi(t)-\psi(0))^{1-\zeta_1}u(t)|_{t=0}=u_0$*;* $u_2\in C_{1-\zeta_2,\psi}([0,T_2])$ *satisfying* (17) *and* $(\psi(t)-\psi(0))^{1-\zeta_2}u(t)|_{t=0}=u_0$*;* $u_i\in C_{1-\zeta_i,\psi}([0,T_i])$ *satisfying* (18) *and* $(\psi(t)-\psi(0))^{1-\zeta_i}u(t)|_{t=0}=u_0$ *for* $(i=3,4,...,N)$.

Where

$$C_{1-\zeta_i,\psi}([0,T_i]) = \{\Phi(t):(0,T_i]\to\mathbb{R},(\psi(t)-\psi(0))^{1-\zeta_i}\Phi(t)\in C^1([0,T_i],\mathbb{R}), \quad (i=1,2,...,N), N\in\mathbb{N}^*\}$$

Lemma 3.1. *Let* $f\in C(J\times\mathbb{R},\mathbb{R}^*)$*,* $J=[0,T]$ *and* $h\in C_{1-\zeta,\psi}(J,\mathbb{R})$*. Then the problem* (3) *has a solution given by*

$$u(t)=f(t,\chi_{\beta_1,u}(t))\left\{\frac{u_0}{f(0,\chi_{\beta_1,u}(0))}(\psi(t)-\psi(0))^{\zeta_1-1}+I_{0_+}^{\alpha_1;\psi}h(t,\chi_{\beta_1,u}(t))\right\}, \tag{19}$$

in $[0,T_1]$.

Again

$$u(t)=f(t,\chi_{\beta_2,u}(t))\left\{\frac{u_0}{f(0,\chi_{\beta_2,u}(0))}(\psi(t)-\psi(0))^{\zeta_2-1}+I_{0_+}^{\alpha_2;\psi}h(t,\chi_{\beta_2,u}(t))\right\}, \tag{20}$$

in $[0,T_2]$.

Repeatly, in the interval $[T_{i-1},T_i]$, $(i=3,4,...,N)$ we have

$$u(t)=f(t,\chi_{\beta_i,u}(t))\left\{\frac{u_0}{f(0,\chi_{\beta_i,u}(0))}(\psi(t)-\psi(0))^{\zeta_i-1}+I_{0_+}^{\alpha_i;\psi}h(t,\chi_{\beta_i,u}(t))\right\}, \tag{21}$$

where $\chi_{\beta_i,u}(t)={}^H D_{0_+}^{\beta_i,\sigma,\psi}u(t)$, $(i=1,2,...,N)$.

Proof. The idea of the proof is the same as that used in [9].

In the interval $(0,T_1]$ the problem (3) can be written by

$$\begin{cases} {}^H D_{0_+}^{\alpha_1,\sigma,\psi}\left(\dfrac{u(t)}{f(t,\chi_{\beta_1,u}(t))}\right) = h(t),\chi_{\beta_1,u}(t)), \quad t\in[0,T_1] \\ (\psi(t)-\psi(0))^{1-\zeta_1}u(t)\mid_{t=0} = u_0, \quad u_0\in\mathbb{R} \end{cases} \tag{22}$$

applying the ψ-Riemann-Liouville fractional integral operator $I_{0_+}^{\alpha_1;\psi}$ on both sides of the problem (22) and using Lemma (2.3), we obtain:

$$\frac{u(t)}{f(t,\chi_{\beta_1,u}(t))} - \frac{(\psi(t)-\psi(0))^{\zeta_1-1}}{\Gamma(\zeta_1)} I_{0_+}^{(1-\zeta_1);\psi}\left(\frac{u(t)}{f(t,\chi_{\beta_1,u}(t))}\right)_{t=0} = I_{0_+}^{\alpha_1;\psi}h(t,\chi_{\beta_1,u}(t)), \quad t\in[0,T_1]$$

this implies that

$$u(t) = f(t,\chi_{\beta_1,u}(t))\left\{\frac{(\psi(t)-\psi(0))^{\zeta_1-1}}{\Gamma(\zeta_1)} I_{0_+}^{(1-\zeta_1);\psi}\left(\frac{u(t)}{f(t,\chi_{\beta_1,u}(t))}\right)_{t=0} + I_{0_+}^{\alpha_1;\psi}h(t,\chi_{\beta_1,u}(t))\right\}, \quad t\in[0,T_1] \tag{23}$$

we pose

$$C = I_{0_+}^{(1-\zeta_1);\psi}\left(\frac{u(t)}{f(t,\chi_{\beta_1,u}(t))}\right)_{t=0}$$

next, we evaluate the value of C using the initial condition. Multiplying $(\psi(t)-\psi(0))^{1-\zeta_1}$ on both sides of the equation (23), we get

$$(\psi(t)-\psi(0))^{1-\zeta_1}u(t) = \frac{C}{\Gamma(\zeta_1)}f(t,\chi_{\beta_1,u}(t)) + (\psi(t)-\psi(0))^{1-\zeta_1}f(t,\chi_{\beta_1,u}(t))I_{0_+}^{\alpha_1;\psi}h(t,\chi_{\beta_1,u}(t)), \quad t\in[0,T_1]$$

putting t = 0 in the above equation and using the initial condition, we obtain

$$C = \Gamma(\zeta_1)\frac{u_0}{f(0,\chi_{\beta_1,u}(0))}$$

putting the value of C in the equation (23) then, we have

$$u(t) = f(t,\chi_{\beta_1,u}(t))\left\{\frac{u_0}{f(0,\chi_{\beta_1,u}(0))}(\psi(t)-\psi(0))^{\zeta_1-1} + I_{0_+}^{\alpha_1;\psi}h(t,\chi_{\beta_1,u}(t))\right\} \quad t\in[0,T_1].$$

On the other hand, suppose that u(t) is the solution of the fractional integral equation (19). Then, it can be written as

$$\left(\frac{u(t)}{f(t,\chi_{\beta_1,u}(t))}\right) = \frac{u_0}{f(0,\chi_{\beta_1,u}(0))}(\psi(t)-\psi(0))^{\zeta_1-1} + I_{0_+}^{\alpha_1;\psi}h(t,\chi_{\beta_1,u}(t)), \quad t\in[0,T_1] \tag{24}$$

applying the $\psi-Hilfer$ fractional derivative operator ${}^H D_{0_+}^{\alpha_1,\sigma,\psi}$ on both sides of the equation (24) and using the Lemmas (2.3) and (2.1)(*ii*), we get:

$${}^H D_{0_+}^{\alpha_1,\sigma,\psi}\left(\frac{u(t)}{f(t,\chi_{\beta_1,u}(t))}\right) = h(t,\chi_{\beta_1,u}(t)), \quad t\in[0,T_1]$$

It remains to verify the initial condition. taking the limit $t \to 0$ of the following equation:

$$(\psi(t)-\psi(0))^{1-\zeta_1}u(t) = (\psi(t)-\psi(0))^{1-\zeta_1} f(t,\chi_{\beta_1,u}(t))\left\{\frac{u_0}{f(0,\chi_{\beta_1,u}(0))}(\psi(t)-\psi(0))^{\zeta_1-1} + I_{0_+}^{\alpha_1;\psi} h(t,\chi_{\beta_1,u}(t))\right\}$$

we get:

$$(\psi(t)-\psi(0))^{1-\zeta_1}u(t)\,|_{t=0} = u_0, \quad u_0 \in \mathbb{R}.$$

In the same way, in the interval $[T_{i-1},T_i]$, $(i=2,3,4,\ldots,N)$ we have

$$u(t) = f(t,\chi_{\beta_i,u}(t))\left\{\frac{u_0}{f(0,\chi_{\beta_i,u}(0))}(\psi(t)-\psi(0))^{\zeta_i-1} + I_{0_+}^{\alpha_i;\psi} h(t,\chi_{\beta_i,u}(t))\right\}, \tag{25}$$

where $\chi_{\beta_i,u}(t) = {}^H D_{0_+}^{\beta_i,\sigma,\psi} u(t)$, $(i=2,3,\ldots,N)$.

The proof is completed.

Next, we introduce the following assumptions:

- The function $f \in C(J\times\mathbb{R},\mathbb{R}^*)$ is bounded and there exists a constant $\delta>0$ such that for all $p,q\in\mathbb{R}$, and $t\in J$ we have: $|f(t,p)-f(t,q)|\le\delta|p-q|$ and L is the constant such that $|f(t,p)|\le L$
- The function $h\in C(J\times\mathbb{R},\mathbb{R})$ and there exists a function $K\in C_{1-\zeta,\psi}(J,\mathbb{R})$ such that:

$$|h(t,p)| \le (\psi(t)-\psi(0))^{1-\zeta}K(t) \quad t\in J, p\in\mathbb{R}$$

- $I_{0_+}^{1-\theta_i;\psi}u(0)=0$, where $\theta_i=\beta_i+\sigma(1-\beta_i)$ for all $(i=1,2,\ldots,N)$
- $\left|\dfrac{u_0}{f(0,\chi_{\beta_i,u}(0))}\right|<1$, where $\chi_{\beta_i,u}(0) = {}^H D_{0_+}^{\beta_i,\sigma,\psi}u(0)$ for all $(i=1,2,3,\ldots,N)$

Let $X := (C_{1-\zeta,\psi}(J,\mathbb{R}), \|\,.\,\|_{C_{1-\zeta,\psi}(J,\mathbb{R})})$. Then X is a Banach algebra. Define,

$$S_i = \left\{v\in X, \|v\|_{C_{1-\zeta_i,\psi}([0,T_i],\mathbb{R})} \le R_i\right\}, \quad (i=1,2,\ldots,N)$$

where :

$$R_i = (\psi(T_i)-\psi(0))^{1-\zeta_i} L\left(\frac{u_0}{f(0,\chi_{\beta_i,u}(0))\delta} + \frac{(\psi(T_i)-\psi(0))^{\alpha_i-\beta_i}}{\Gamma(\alpha_i-\beta_i+1)}\|K\|_{C_{1-\zeta_i,\psi}([0,T_i],\mathbb{R})}\right), \quad (i=1,2,\ldots,N)$$

Clearly, S_i is a closed, convex and nonempty subset of X for all $(i=1,2,\ldots,N)$.

Theorem 3.1. *Assume that the assumptions* (A_1), (A_2), (A_3), *and* (A_4) *hold. Then, the system of nonlinear ψ-Hilfer hybrid FDE* (3) *has a solution* $u\in C_{1-\zeta,\psi}(J,\mathbb{R})$ *provided:*

$$I_{0_+}^{-\beta_i;\psi}(\psi(t)-\psi(0))^{\zeta_i-1} < \frac{1}{\delta}, \quad \text{for } (i=1,2,\ldots,N) \tag{26}$$

Proof. To obtain our main results, we start by performing the essential analysis of the problem (3).

Let $v(t) = {}^H D_{0_+}^{\beta_i,\sigma,\psi}u(t))$ for all $(i=1,2,\ldots,N)$, then according to the assumption (A_3) and the lemma (2.3), we obtain

$$u(t) = I_{0_+}^{\beta_i;\psi} v(t)$$

Then from the equation (21) we will consider the solution of the following integral equation

$$v(t)=\frac{u_0}{f(0,v(0))}I_{0_+}^{-\beta_i;\psi}f(t,v(t))(\psi(t)-\psi(0))^{\zeta_i-1}+I_{0_+}^{-\beta_i;\psi}f(t,v(t))I_{0_+}^{\alpha_i;\psi}h(t,v(t)),\quad t\in[0,T_i],\quad (i=1,2,\ldots,N). \quad (27)$$

Obviously, if $v^*\in C_{1-\zeta_i,\psi}([0,T_i])$ is a solution of (27), then, applying fractional integral operator $I_{0_+}^{\beta_i;\psi}$ on both sides of (27), from the lemma (2.2), we get

$$\begin{aligned}I_{0_+}^{\beta_i;\psi}v^*(t)&=I_{0_+}^{\beta_i;\psi}\frac{u_0}{f(0,v^*(0))}I_{0_+}^{-\beta_i;\psi}f(t,v^*(t))(\psi(t)-\psi(0))^{\zeta_i-1}+I_{0_+}^{\beta_i;\psi}I_{0_+}^{-\beta_i;\psi}f(t,v^*(t))I_{0_+}^{\alpha_i;\psi}h(t,v^*(t))\\&=f(t,v^*(t))\left\{\frac{u_0}{f(0,v^*(0))}(\psi(t)-\psi(0))^{\zeta_i-1}+I_{0_+}^{\alpha_i;\psi}h(t,v^*(t))\right\},\quad t\in[0,T_i]\end{aligned}$$

Let $u^*(t)=I_{0_+}^{\beta_i;\psi}v^*(t)$, then we have

$$u^*(t)=f(t,v^*(t))\left\{\frac{u_0}{f(0,v^*(0))}(\psi(t)-\psi(0))^{\zeta_i-1}+I_{0_+}^{\alpha_i;\psi}h(t,v^*(t))\right\}$$

that is, u^* is a solution of (21), with the condition $(\psi(t)-\psi(0))^{1-\zeta_i}u^*(t)|_{t=0}=u_0$, for all $(i=1,2,\ldots,N)$.

Define the operators $A:X\to X$ and $B:S_1\to X$, for $t\in[0,T_1]$ by:

$$\begin{aligned}Av(t)&=\frac{u_0}{f(0,v(0))}I_{0_+}^{-\beta_1;\psi}f(t,v(t))(\psi(t)-\psi(0))^{\zeta_1-1}\\Bv(t)&=I_{0_+}^{-\beta_1;\psi}f(t,v(t))I_{0_+}^{\alpha_1;\psi}h(t,v(t)).\end{aligned}$$

We consider the mapping $T:S_1\to X$ defined by:

$$Tv(t)=Av(t)+Bv(t),\quad t\in[0,T_1]$$

To prove that $u\in C_{1-\zeta_1,\psi}([0,T_1],\mathbb{R})$ is a solution to the problem (3) is equivalent to proving that the mapping T has a fixed point, we show that the operators A and B satisfy the conditions of the Theorem (2.1).

The proof is given in several steps:

Step 1: Let $v,y\in S_1$, then for all $t\in[0,T_1]$ according to the assumptions A_1, A_2 and the condition (26) we have:

$$\begin{aligned}|(\psi(t)-\psi(0))^{1-\zeta_1}(Av(t)+By(t))|&\le\left|(\psi(t)-\psi(0))^{1-\zeta_1}\frac{u_0}{f(0,v(0))}I_{0_+}^{-\beta_1;\psi}f(t,v(t))(\psi(t)-\psi(0))^{\zeta_1-1}\right|\\&+|(\psi(t)-\psi(0))^{1-\zeta_1}I_{0_+}^{-\beta_1;\psi}f(t,y(t))I_{0_+}^{\alpha_1;\psi}h(t,y(t))|\\&\le(\psi(T_1)-\psi(0))^{1-\zeta_1}\frac{u_0}{f(0,v(0))}I_{0_+}^{-\beta_1;\psi}|f(t,v(t))|(\psi(t)-\psi(0))^{\zeta_1-1}\\&+(\psi(T_1)-\psi(0))^{1-\zeta_1}LI_{0_+}^{\alpha_1-\beta_1;\psi}|h(t,y(t))|\\&\le\frac{u_0(\psi(T_1)-\psi(0))^{1-\zeta_1}L}{f(0,v(0))\delta}+\frac{(\psi(T_1)-\psi(0))^{\alpha_1-\beta_1+1-\zeta_1}L}{\Gamma(\alpha_1-\beta_1+1)}\|K\|_{C_{1-\zeta_1,\psi}([0,T_1],\mathbb{R})}\\&\le(\psi(T_1)-\psi(0))^{1-\zeta_1}L\left(\frac{u_0}{f(0,v(0))\delta}+\frac{(\psi(T_1)-\psi(0))^{\alpha_1-\beta_1}}{\Gamma(\alpha_1-\beta_1+1)}\|K\|_{C_{1-\zeta_1,\psi}([0,T_1],\mathbb{R})}\right)\end{aligned}$$

this gives:

$$\| (Av(t)+By(t)) \|_{C_{1-\zeta_1,\psi}([0,T_1],\mathbb{R})} \leq R_1$$

this implies, $(Av(t)+By(t)) \in S_1$ for all $v, y \in S_1$.

Step 2: A is a contraction on S_1. Clearly from Assumption A_1 and (A_4), we have

$$\begin{aligned}
&|(\psi(t)-\psi(0))^{1-\zeta_1}(Av(t)-Ay(t))| \\
&= \left|(\psi(t)-\psi(0))^{1-\zeta_1} \frac{u_0}{f(0,v(0))} I_{0_+}^{-\beta_1;\psi} (\psi(t)-\psi(0))^{\zeta_1-1}(f(t,v(t))-f(t,y(t)))\right| \\
&\leq (\psi(t)-\psi(0))^{1-\zeta_1} \left|\frac{u_0}{f(0,v(0))}\right| \frac{1}{\Gamma(-\beta_1)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{-\beta_1-1}(\psi(s)-\psi(0))^{\zeta_1-1} |(f(s,v(s))-f(s,y(s)))|\,ds \\
&\leq \delta I_{0_+}^{-\beta_1;\psi}(\psi(t)-\psi(0))^{\zeta_1-1} \sup_{t\in[0,T_1]} |(\psi(t)-\psi(0))^{1-\zeta_1}(v(t)-y(t))| \\
&\leq \delta I_{0_+}^{-\beta_1;\psi}(\psi(t)-\psi(0))^{\zeta_1-1} \| v-y \|_{C_{1-\zeta_1,\psi}([0,T_1],\mathbb{R})}.
\end{aligned}$$

Then, according to the condition (26) and Lemma (2.5) the operator A is a contraction on S_1.

Step 3: $B: S \to X$ is completely continuous:

- $B: S_1 \to X$ is continuous
  Let $(v_n)_{n\in\mathbb{N}}$ be any sequence in S_1 such that $v_n \to v$ as $n \to \infty$ in S_1. We prove that $Bv_n \to Bv$ as $n \to \infty$ in S_1. We have

$$\begin{aligned}
\| Bv_n - Bv \|_{C_{1-\zeta_1,\psi}([0,T_1],\mathbb{R})} &= |(\psi(t)-\psi(0))^{1-\zeta_1}(I_{0_+}^{-\beta_1;\psi} f(t,v_n(t)) I_{0_+}^{\alpha_1;\psi} h(t,v_n(t)) \\
&\quad - I_{0_+}^{-\beta_1;\psi} f(t,v(t)) I_{0_+}^{\alpha_1;\psi} h(t,v(t))| \\
&\leq (\psi(T_1)-\psi(0))^{1-\zeta_1} L\, | I_{0_+}^{\alpha_1-\beta_1;\psi}(h(t,v_n(t))-h(t,v(t)))|
\end{aligned}$$

  By Lebesgue dominated convergence theorem, from the above inequality, we obtain:

$$\| Bv_n - Bv \|_{C_{1-\zeta_1,\psi}([0,T_1],\mathbb{R})} \to 0 \text{ as } n \to \infty.$$

  This proves that $B: S_1 \to X$ is continuous.

- $B(S_1) = \{Bv : v \in S_1\}$ is uniformly bounded.
  Using assumptions (A_1) and (A_2), for any $v \in S_1$ and $t \in [0, T_1]$, we have:

$$\begin{aligned}
|(\psi(t)-\psi(0))^{1-\zeta_1} Bv(t)| &= |(\psi(t)-\psi(0))^{1-\zeta_1} I_{0_+}^{-\beta_1;\psi} f(t,v(t)) I_{0_+}^{\alpha_1;\psi} h(t,v(t))| \\
&\leq (\psi(T_1)-\psi(0))^{1-\zeta_1} L I_{0_+}^{\alpha_1-\beta_1;\psi} |h(t,y(t))| \\
&\leq \frac{(\psi(T_1)-\psi(0))^{\alpha_1-\beta_1+1-\zeta_1} L}{\Gamma(\alpha_1-\beta_1+1)} \| K \|_{C_{1-\zeta_1,\psi}([0,T_1],\mathbb{R})},
\end{aligned}$$

  therfore

$$\| Bv \|_{C_{1-\zeta_1,\psi}([0,T_1],\mathbb{R})} \leq \frac{(\psi(T_1)-\psi(0))^{\alpha_1-\beta_1+1-\zeta_1} L}{\Gamma(\alpha_1-\beta_1+1)} \| K \|_{C_{1-\zeta_1,\psi}([0,T_1],\mathbb{R})} \tag{28}$$

this implies that $B(S_1)=\{Bv : v\in S_1\}$ is uniformly bounded.

- $B(S_1)$ is equicontinuous
  Let any $v\in S_1$ and $t_1,t_2\in J$ with $t_1<t_2$, then using hypothesis (A_1) and (A_2), we have

$$
\begin{aligned}
&|(\psi(t_2)-\psi(0))^{1-\zeta_1}Bv(t_2)-(\psi(t_1)-\psi(0))^{1-\zeta_1}Bv(t_1)|\\
&=|(\psi(t_2)-\psi(0))^{1-\zeta_1}I_{0_+}^{-\beta_1;\psi}f(t_2,v(t_2))I_{0_+}^{\alpha_1;\psi}h(t_2,v(t_2))-(\psi(t_1)-\psi(0))^{1-\zeta_1}I_{0_+}^{-\beta_1;\psi}f(t_1,v(t_1))I_{0_+}^{\alpha_1;\psi}h(t_1,v(t_1))|\\
&\le|(\psi(t_2)-\psi(0))^{1-\zeta_1}\frac{1}{\Gamma(-\beta_1)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{-\beta_1-1}|f(s,v(s))|\,I_{0_+}^{\alpha_1;\psi}|h(s,v(s))|\,ds\\
&-(\psi(t_1)-\psi(0))^{1-\zeta_1}\frac{1}{\Gamma(-\beta_1)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{-\beta_1-1}|f(s,v(s))|\,I_{0_+}^{\alpha_1;\psi}|h(s,v(s))|\,ds|\\
&\le L|(\psi(t_2)-\psi(0))^{1-\zeta_1}\frac{1}{\Gamma(\alpha_1-\beta_1)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha_1-\beta_1-1}|h(s,v(s))|\,ds\\
&-(\psi(t_1)-\psi(0))^{1-\zeta_1}\frac{1}{\Gamma(\alpha_1-\beta_1)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\alpha_1-\beta_1-1}|h(s,v(s))|\,ds|\\
&\le\frac{L\,\|K\|_{C_{1-\zeta_1,\psi}([0,T_1],\mathbb{R})}}{\Gamma(\alpha_1-\beta_1+1)}|(\psi(t_2)-\psi(0))^{\alpha_1-\beta_1+1-\zeta_1}-(\psi(t_1)-\psi(0))^{\alpha_1-\beta_1+1-\zeta_1}|
\end{aligned}
$$

by the continuity of ψ, from the above inequality it follows that:

If $|t_1-t_2|\mapsto 0$ then $|(\psi(t_2)-\psi(0))^{1-\zeta_1}Bv(t_2)-(\psi(t_1)-\psi(0))^{1-\zeta_1}Bv(t_1)|\mapsto 0$ From the parts (ii) and (iii), it follows that $B(S_1)$ is uniformly bounded and equicontinuous set in X. Then by Arzela-Ascoli theorem, $B(S_1)$ is relatively compact. Since $B:S_1\to X$ is the continuous and compact operator, it is completely continuous.

So from steps 1 to 3, it follows that all the conditions of Theorem (2.1) are fulfilled. Hence operator T has a solution in S_1.

This implies that the system of hybrid FDE (3) has a solution in $C_{1-\zeta_1,\psi}([0,T_1],\mathbb{R})$.

As a result, according to the arguments above, $u(t)=I_{0^+}^{\beta_1}v(t)$ is a solution to the problem (3). In a similar way, for all $i=2,3,...N,N$, $(T_N=T)$, we obtain that the problem (3) has a solution $u(t)=I_{0^+}^{\beta_i}v(t)$ in $C_{1-\zeta_i,\psi}([0,T_i],\mathbb{R})$.

4. Example

In this section, we give an example that is relevant to demonstrate our results. We consider the particular case when $\psi(t)=t$ and $\sigma=1$.

Consider the ψ-Hilfer hybrid FDE involving Caputo fractional derivative.

Where $f(t,{}^CD_{0_+}^{\beta(t)}u(t))=\frac{1}{2}+\frac{1}{4}\cos({}^CD_{0_+}^{\beta(t)}u(t))$, $h(t,{}^CD_{0_+}^{\beta(t)}u(t))=\frac{{}^CD_{0_+}^{\beta(t)}u(t)}{1+({}^CD_{0_+}^{\beta(t)}u(t))^2}$, $J=[0,3]$, and

$$
\alpha(t)=\begin{cases}\alpha_1=\frac{1}{2}, & t\in[0,1]\\ \alpha_2=\frac{2}{3}, & t\in[1,2]\\ \alpha_3=\frac{3}{4}, & t\in[2,3]\end{cases}
$$

$$\beta(t)=\begin{cases}\beta_1=\frac{1}{4}, & t\in[0,1]\\ \beta_2=\frac{2}{4}, & t\in[1,2]\\ \beta_3=\frac{3}{5}, & t\in[2,3]\end{cases}$$

$$\begin{cases}{}^{C}D^{\alpha(t)}\left(\dfrac{u(t)}{\frac{1}{2}+\frac{1}{4}\cos({}^{C}D_{0_+}^{\beta(t)}u(t))}\right)=\dfrac{{}^{C}D_{0_+}^{\beta(t)}u(t)}{1+({}^{C}D_{0_+}^{\beta(t)}u(t))^2}, t\in[0,3]\\ u(0)=0\end{cases} \tag{29}$$

Let $v(t)={}^{C}D_{0_+}^{\beta(t)}u(t)$, then $f(t,v(t))=\frac{1}{2}+\frac{1}{4}\cos(v(t))$, $h(t,v(t))=\dfrac{v(t)}{1+(v(t))^2}$,

it is clear that

$|f(t,v(t))-f(t,y(t))|\leq\frac{1}{4}|v-y|$, $|f(t,v(t))|\leq\frac{3}{4}$, $\left|\dfrac{u_0}{f(0,v(0))}\right|=0<1$, and $|h(t,v(t))|\leq 1$, here $k(t)=1$, $L=\frac{3}{4}$, and $\delta=\frac{1}{4}$.

We have $I_{0_+}^{1-\theta_i;\psi}u(0)=I_{0_+}^{0}u(0)=0$, where $\theta_i=\beta_i+\sigma(1-\beta_i)$ for all $(i=1,2,3)$.

Now we check for condition (26). Then for all (i = 1, 2, 3), we have

$$\begin{aligned}I_{0_+}^{-\beta_i;\psi}(\psi(t)-\psi(0))^{\zeta_i-1}&=I_{0_+}^{-\beta_i}(t-0)^{1-1}\\&=I_{0_+}^{-\beta_i}1\\&\leq\frac{T_i^{-\beta_i}}{\Gamma(-\beta_i+1)},\quad(T_1=1,T_2=2,T_3=3)\\&<\frac{1}{\delta}=4\end{aligned}$$

we observe that all the conditions of Theorem (3.1) are satisfied. Therefore, the system ψ-Hilfer hybrid FDE involving Caputo fractional derivative (29) has a solution $u(t)=I_{0^+}^{\beta(t)}v(t)$ in $C([0,3])$, such that

$$u(t)=\begin{cases}u_1(t)=I_{0^+}^{\beta_1}v_1(t)=\left(\frac{1}{2}+\frac{1}{4}\sin(v_1(t))\right)I_{0_+}^{\frac{1}{2}}\left(\dfrac{v_1(t))}{1+v_1(t)^2}\right), & t\in[0,1]\\ u_2(t)=I_{0^+}^{\beta_2}v_2(t)=\left(\frac{1}{2}+\frac{1}{4}\sin(v_2(t))\right)I_{0_+}^{\frac{2}{3}}\left(\dfrac{v_2(t))}{1+v_2(t)^2}\right), & t\in[1,2]\\ u_3(t)=I_{0^+}^{\beta_3}v_3(t)=\left(\frac{1}{2}+\frac{1}{4}\sin(v_3(t))\right)I_{0_+}^{\frac{3}{4}}\left(\dfrac{v_3(t))}{1+v_3(t)^2}\right), & t\in[2,3]\end{cases}$$

where $v_i(t)={}^{C}D_{0_+}^{\beta_i}u_i(t)$, for all $(i=1,2,3)$

5. Conclusion

This work introduces the concepts of variable order fractional integration and differentiation where the order is a function of time t. We have established existence theory on solutions of the system of hybrid fractional differential equation of variable order involving ψ-Hilfer fractional derivative by using the Krasnoselskii fixed point theorem. Also, we presented an example to illustrate our main results.

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