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Common best proximity points of some graph-theoretical notions of dominating pairs

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Abstract

This article deals with a metric space endowed with a directed graph. In order to obtain common best proximity point results, we introduce a concept of Geraghty's type contraction satisfying certain properties on the graph. An example and some consequences are also provided.

Keywords: best proximity point; common best proximity point; Geraghty's type contraction; G–proximal.

Mathematics Subject Classification: 41A17, 47H09, 47H10

1. Introduction

Let (X, d) be a metric space, and A, B be nonempty subsets of X. The existence of a solution to the equation $Tx = x$, where $T : A \to B$, is subject to the mapping T itself and depends on the sets A, B. For instance, If A and B are disjoint, then $Tx = x$ never has a solution, a *fixed point*. This leads to investigation of a more general equation

$$d(x^*, Tx^*) = d(A, B),$$

where the distance is involved. A solution x^* to this equation is called a *best proximity point*. It is clearly seen that if $d(A, B) = 0$, then a best proximity point becomes a fixed point.

Theorems regarding best proximity points were first studied in [11]. They have been generalized in various aspects as well as applications in real world problems. Karapnar have given contributions to

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the theory establishing a number of remarkable results; for example in [18, 14, 15, 16, 20, 19]. Many authors have also contributed assertions concerning best proximity points; see [1, 5, 22, 10, 25, 26, 29], to mention but a few given.

Let $S : A \to B$ be another nonself mapping. Again, the system of the equations $Tx = x$ and $Sx = x$ does not need to have a solution. One may, instead, deal with a more general system

$$d(x^*, Tx^*) = d(A, B) = d(x^*, Sx^*).$$

A solution to this system is known as a *common best proximity point* of S and T.

A result concerning common best proximity points was first studied in [3], followed by Kuman's and Mongkolkeha's work establishing theorems for proximity commuting Geraghty's type mappings, see [24]. Later, Chen [7] created a class of pairs of mapping (P, Q) where Q is proximally dominated by P, and obtained some common best proximity point results. The results were then discovered by introducing a type of Geraghty contractions in [21].

Jachymski [13] initiated the concept of fixed points regarding a metric space endowed with graph G which generalizes the Banach contraction principle. Recently, the notion of a G–proximal that generalized contraction were presented as well as several best proximity point results in [23]. There have been many research articles dealing with this concept. The reader may be referred to [8, 30, 6, 28, 2, 12, 27, 9], for example.

In this article, we consider a metric space (X, d) which is endowed with graph G. Motivated by the work in [21], we provide a class of pairs of mappings in X associated with auxiliary functions introduced in [17]. Then, some sufficient conditions for the existence and uniqueness of a common best proximity point in X are presented. We also give an example and show some consequences of our main results.

2. Preliminaries

Let (X, d) be a metric space. For a pair of nonempty subsets A and B of X, the distance between them is defined by

$$d(A, B) = \inf\{d(x, y) : x \in A \;\; and \;\; y \in B\}.$$

Notice that the $d(A, B)$ always exists; and, in particular, if A and B are closed disjoint subsets, then the their distance is guaranteed to be positive.

Throughout this section, let f and g be any mappings between nonempty subsets A, B of a metric space (X, d).

Definition 2.1. [24] An $x^* \in A$ is said to be a *common best proximity point* of f and g if

$$d(x^*, fx^*) = d(A, B) = d(x^*, gx^*).$$

The set of such points is denoted by $\mathcal{CB}(f, g)$.

In fact, if $x^* \in \mathcal{CB}(f, g)$ with $d(A, B) = 0$, then x^* becomes a *common fixed point* of self-mappings f and g on $A \bigcap B$.

Let us now introduce a more general notion of commutativity of two mappings that becomes useful in this work.

Definition 2.2. [4] A pair of mappings (f, g) is said to *commute proximally* if

$$d(u, fx) = d(v, gx) = d(A, B) \text{ implies } fv = gu.$$

for all $x, u, v \in A$.

Despite being less intuitive, one may view this notion as a generalized commutative property. More precisely, if $fg = gf$ and $d(A, B) = 0$, it follows that $u = fx$ and $v = gx$ yielding $fv = gu$. Notice also that if (f, g) commutes proximally, so does (g, f).

One can add a graph-theoretical structure on X. Namely, X is said to be *endowed with a directed graph* $G=(V_G,E_G)$ if the set of vertices V_G is X itself and the set of edges E_G contains all loops $\{(x,x): x\in X\}$.

Let us begin with pointwise-continuity with respect to G.

Definition 2.3. [13] Let X be endowed with a directed graph G. A mapping $T: X\to X$ is said to be *G–continuous* at $x\in X$ if for any sequence $\{x_n\}$ in X with $(x_n,x_{n+1})\in E_G$ that converges to x, the sequence $\{Tx_n\}$ converges to Tx.

G–continuity allows the interplay between the metric space (X,d) and the graph G. In fact, continuity with respect to d implies continuity with respect to G; and both notions are equivalent if $E(G)=X\times X$.

Definition 2.4. Let X be endowed with a directed graph G. A pair of mappings (f,g) is said to be *G–proximally commutative edge preserving* if for any $x,y,u,v\in A$,

1. $(fx,fy)\in E_G$ implies $(gx,gy)\in E_G$; and
2. $(fx,gx)\in E_G$ and $d(u,gx)=d(A,B)=d(v,fx)$ imply $(v,u)\in E_G$ and $fu=gv$.

The definition above combines proximal commutativity with edge preservation. In fact, if (f,g) is G–proximally commutative edge preserving, then it commutes proximally; however, (g,f) may not be G–proximally commutative edge preserving.

Last but not least, let us introduce a class Γ containing functions $\gamma: X\times X\to[0,1]$ such that for any sequences $\{x_n\}$ and $\{y_n\}$ in X,

$$\lim_{n\to\infty}\gamma(x_n,y_n)=1 \text{implies} \lim_{n\to\infty}d(x_n,y_n)=0,$$

whenever $\{d(x_n,y_n)\}$ is a decreasing real sequence. Let Γ be a class of functions. This class of these auxiliary functions Γ was introduced in [17]. Note that here we also include 1 to the ranges of these functions γ.

We also denote the class of mappings

$$\Psi=\{\psi:[0,\infty)\to[0,\infty): \psi \text{ is continuous, increasing, } t\le\psi(t) \text{ and } \psi(0)=0\}.$$

For $u_1,u_2,v_1,v_2\in X$, we denote

$$M(v_1,v_2,u_1,u_2)=\max\left\{d(v_1,v_2),d(v_1,u_1),d(v_2,u_2),\frac{d(v_1,u_2)+d(v_2,u_1)}{2}\right\}.$$

Definition 2.5. A pair (f,g) is said to be ***G*–proximally dominating** if there exist $\gamma\in\Gamma$ and $\psi\in\Psi$ such that for each $x_1,x_2,u_1,u_2,v_1,v_2\in A$,

$$\psi(d(u_1,u_2))\le\gamma(v_1,v_2)\psi(M(v_1,v_2,u_1,u_2)) \tag{1}$$

whenever $(fx_1,gx_1)\in E_G$, $(fx_2,gx_2)\in E_G$ and

$$d(u_1,gx_1)=d(u_2,gx_2)=d(A,B)=d(v_1,fx_1)=d(v_2,fx_2).$$

3. Main Results

Throughout this section, let X be a complete metric space endowed with a directed graph $G=(V_G,E_G)$, and let f and g be mappings from a nonempty set $A\subseteq X$ to a nonempty set $B\subseteq X$. Moreover, the pair (f,g) will be assumed to be G–proximally dominating and G–proximally commutative edge preserving. Also, let us define

$$A_0=\{x\in A: \text{there exists } y\in B \text{ such that } d(x,y)=d(A,B)\}$$
$$B_0=\{y\in B: \text{there exists } x\in A \text{ such that } d(x,y)=d(A,B)\}$$

that contain all elements of A and B, respectively, minimizing the distance. The sets A_0 and B_0 may be empty in general. However, $A_0 \neq \emptyset$ if and only if $B_0 \neq \emptyset$. For convenience, let us here assume that $A_0 \neq \emptyset$ and $g(A_0) \subseteq B_0$.

Our main purpose is to establish situations where the existence of a common best proximity point of (f,g) is guaranteed. Here, denote by $\mathcal{CB}(f,g)$ the set of common best proximity points of the pair (f,g).

Lemma 3.1. *If there exists $u \in A_0$ such that u is a coincidence point of f and g (i.e., $fu = gu$), then $\mathcal{CB}(f,g) \neq \emptyset$.*

Proof. Let $u \in A_0$ satisfying $fu = gu$. Since $g(A_0) \subseteq B_0$, there exists $x^* \in A_0$ such that

$$d(x^*, fu) = d(x^*, gu) = d(A,B). \tag{2}$$

By the assumption that (f,g) is G–proximally commutative edge preserving, we have $fx^* = gx^*$. Again, since $g(A_0) \subseteq B_0$, there exists $z^* \in A_0$ such that

$$d(z^*, fx^*) = d(z^*, gx^*) = d(A,B). \tag{3}$$

It follows from (2)–(3) that

$$d(x^*, gu) = d(z^*, gx^*) = d(A,B) = d(x^*, fu) = d(z^*, fx^*).$$

Next, we will prove that $x^* = z^*$. Suppose that $d(x^*, z^*) > 0$. Then, $\psi(d(x^*, z^*)) > 0$ and

$$M(x^*, z^*, x^*, z^*) = \max\{d(x^*, z^*), d(x^*, x^*), d(z^*, z^*)\} = d(x^*, z^*).$$

Since $(fu, gu) = (fu, fu) \in E_G$ and $(fx^*, gx^*) = (fx^*, fx^*) \in E_G$ and that (f,g) is G–proximally dominating,

$$\begin{aligned} \psi(d(x^*, z^*)) &\leq \gamma(x^*, z^*)\psi(M(x^*, z^*, x^*, z^*)) \\ &= \gamma(x^*, z^*)\psi(d(x^*, z^*)) \\ &\leq \psi(d(x^*, z^*)). \end{aligned}$$

Since $\psi(d(x^*, z^*)) > 0$, we have that $1 \leq \gamma(x^*, z^*) \leq 1$ and thus $\gamma(x^*, z^*) = 1$. By the property of γ, $d(x^*, z^*) = 0$ which is a contradiction. Therefore, $x^* = z^*$ and thus

$$d(x^*, fx^*) = d(A,B) = d(x^*, gx^*),$$

as required.

Theorem 3.2. *Suppose that the following conditions are satisfied:*

1. A_0 is closed and $g(A_0) \subseteq f(A_0)$;
2. there is $x_0 \in A_0$ such that $(fx_0, gx_0) \in E_G$; and
3. f and g are G–continuous.

Then, $\mathcal{CB}(f,g) \neq \emptyset$. Moreover, if $(fx, gx) \in E_G$ for all $x \in \mathcal{CB}(f,g)$, then (f,g) has a unique common best proximity point.

Proof. From $gx_0 \in g(A_0) \subseteq f(A_0)$, there exists x_1 such that $gx_0 = fx_1$. Thus $(fx_0, fx_1) = (fx_0, gx_0) \in E_G$. Since (f,g) is G–proximally commutative edge preserving, $(gx_0, gx_1) = (fx_1, gx_1) \in E_G$. From $gx_1 \in g(A_0) \subseteq f(A_0)$, there exists x_2 such that $gx_1 = fx_2$. Similarly, $(gx_1, gx_2) = (fx_2, gx_2) \in E_G$. Continuing in this way, we obtain the sequence $\{x_n\}$ in A_0 satisfying

$$gx_n = fx_{n+1} \text{ and } (fx_n, gx_n) \in E_G \text{ for all } n \geq 0. \tag{4}$$

Also, since $g(A_0) \subseteq B_0$, there exists $u_n \in A_0$ such that $d(u_n, gx_n) = d(A,B)$ for all $n \geq 0$. Then, we obtain the sequence $\{u_n\}$ in A_0 satisfying

$$d(u_n, gx_n) = d(A,B) = d(u_n, fx_{n+1}) \text{ for all } n \geq 0. \tag{5}$$

In the case that there exists $n_0 \geq 0$ such that $u_{n_0} = u_{n_0+1} \in A_0$, by (5), we have that

$$d(u_{n_0+1}, gx_{n_0+1}) = d(A,B) = d(u_{n_0}, fx_{n_0+1}).$$

Since(f,g) is G–proximally commutative edge preserving,

$$g(u_{n_0}) = f(u_{n_0+1}) = f(u_{n_0}).$$

It follows from Lemma 3.1 that $\mathcal{CB}(f,g) \neq \emptyset$.

We suppose that $u_n \neq u_{n+1}$ for all $n \geq 0$. From (5), we note that for all $n \geq 1$

$$d(u_n, gx_n) = d(A,B) = d(u_{n-1}, fx_n) \tag{6}$$

and

$$d(u_{n+1}, gx_{n+1}) = d(A,B) = d(u_n, fx_{n+1}). \tag{7}$$

From $(fx_n, gx_n) \in E_G$, $(fx_{n+1}, gx_{n+1}) \in E_G$, (6), (7) and (f,g) being G–proximally dominating, we deduce that

$$\begin{aligned}\psi(d(u_n,u_{n+1})) &\leq \gamma(u_{n-1},u_n)\psi(M(u_{n-1},u_n,u_n,u_{n+1})) \\ &\leq \psi(M(u_{n-1},u_n,u_n,u_{n+1}))\end{aligned} \tag{8}$$

where

$$\begin{aligned}M(u_{n-1},u_n,u_n,u_{n+1}) &= \max\left\{d(u_{n-1},u_n), d(u_n,u_{n+1}), \frac{d(u_{n-1},u_{n+1})}{2}\right\} \\ &\leq \max\left\{d(u_{n-1},u_n), d(u_n,u_{n+1}), \frac{d(u_{n-1},u_n)+d(u_n,u_{n+1})}{2}\right\} \\ &= \max\{d(u_{n-1},u_n), d(u_n,u_{n+1})\} \leq M(u_{n-1},u_n,u_n,u_{n+1}).\end{aligned}$$

This implies that $M(u_{n-1},u_n,u_n,u_{n+1}) = \max\{d(u_{n-1},u_n), d(u_n,u_{n+1})\}$ for all $n \geq 1$.

Next, we claim that the sequence $\lim_{n\to\infty} d(u_n,u_{n+1}) = 0$. Consider the following two cases.

If $M(u_{n-1},u_n,u_n,u_{n+1}) = d(u_{n-1},u_n)$, then from (8), we have

$$\psi(d(u_n,u_{n+1})) \leq \gamma(u_{n-1},u_n)\psi(d(u_{n-1},u_n) \leq \psi(d(u_{n-1},u_n) \tag{9}$$

for all $n \geq 1$. Therefore, $\{\psi(d(u_n,u_{n+1}))\}$ is nonincreasing. By the property of ψ, we have that $\{(d(u_n,u_{n+1})\}$ is nonincreasing and bounded below. Thus, the sequence is convergent. Suppose that $\lim_{n\to\infty} d(u_n,u_{n+1}) > 0$. Then, $\lim_{n\to\infty}\psi(d(u_n,u_{n+1})) > 0$. By (9), we obtain that

$$1 = \lim_{n\to\infty}\frac{\psi(d(u_n,u_{n+1}))}{\psi(d(u_{n-1},u_n))} \leq \lim_{n\to\infty}\gamma(u_{n-1},u_n) \leq 1.$$

This implies that $\lim_{n\to\infty}\gamma(u_{n-1},u_n) = 1$. By the definition of γ, $\lim_{n\to\infty} d(u_{n-1},u_n) = 0$ which is a contradiction.

If $M(u_{n-1},u_n,u_n,u_{n+1}) = d(u_n,u_{n+1})$, similarly by (8), we have that

$$\psi(d(u_n,u_{n+1})) \leq \gamma(u_{n-1},u_n)\psi(d(u_n,u_{n+1})) \leq \psi(d(u_n,u_{n+1})).$$

Since $d(u_n,u_{n+1})>0$ for all $n\geq 0$, then $\psi(d(u_n,u_{n+1}))>0$ and so $\lim_{n\to\infty}\gamma(u_{n-1},u_n)=1$. It follows that

$$\lim_{n\to\infty} d(u_{n-1},u_n)=0 \tag{10}$$

as required.

Next, we claim that $\{u_n\}$ is a Cauchy sequence. Suppose for a contradiction. Then, there exists $\varepsilon>0$ such that for all $k\in\mathbb{N}$, there are $m_k>n_k>k$ satisfying

$$d(u_{m_k},u_{n_k})\geq\varepsilon. \tag{11}$$

Note that we can choose the smallest n_k satisfying (11) for all $k\in\mathbb{N}$. Thus,

$$d(u_{m_k},u_{n_k-1})<\varepsilon. \tag{12}$$

By (11) and (12), we have that

$$\varepsilon\leq d(u_{m_k},u_{n_k})\leq d(u_{m_k},u_{n_k-1})+d(u_{n_k-1},u_{n_k})<\varepsilon+d(u_{n_k-1},u_{n_k}). \tag{13}$$

From (10), we subsequently have that $\lim_{k\to\infty} d(u_{m_k},u_{n_k})=\varepsilon$. Then, by the triangle inequality,

$$d(u_{m_k},u_{n_k})\leq d(u_{m_k},u_{m_k+1})+d(u_{m_k+1},u_{n_k+1})+d(u_{n_k+1},u_{n_k}).$$

This implies that $\varepsilon=\lim_{k\to\infty}d(u_{m_k},u_{n_k})\leq\lim_{k\to\infty}d(u_{m_k+1},u_{n_k+1})$. Similarly, since

$$d(u_{m_k+1},u_{n_k+1})\leq d(u_{m_k+1},u_{m_k})+d(u_{m_k},u_{n_k})+d(u_{n_k},u_{n_k+1}),$$

$\lim_{k\to\infty}d(u_{m_k+1},u_{n_k+1})\leq\lim_{k\to\infty}d(u_{m_k},u_{n_k})=\varepsilon$. Therefore,

$$\lim_{k\to\infty} d(u_{m_k},u_{n_k})=\lim_{k\to\infty} d(u_{m_k+1},u_{n_k+1})=\varepsilon. \tag{14}$$

We note from (5) that

$$\begin{aligned} d(u_{n_k+1},gx_{n_k+1})&=d(A,B)=d(u_{n_k},fx_{n_k+1}),\text{ and}\\ d(u_{m_k+1},gx_{m_k+1})&=d(A,B)=d(u_{m_k},fx_{m_k+1}) \end{aligned} \tag{15}$$

for all $k\geq 1$.

Since (4), $(fx_{n_k+1},gx_{n_k+1})\in E_G$, $(fx_{m_k+1},gx_{m_k+1})\in E_G$, (15) and (f,g) is G–proximally dominating,

$$\psi(d(u_{n_k+1},u_{m_k+1}))\leq\gamma(u_{n_k},u_{m_k})\psi(M(u_{n_k},u_{m_k},u_{n_k+1},u_{m_k+1})) \tag{16}$$

where

$$\begin{aligned} &M(u_{n_k},u_{m_k},u_{n_k+1},u_{m_k+1})\\ &=\max\left\{d(u_{n_k},u_{m_k}),d(u_{n_k},u_{n_k+1}),d(u_{m_k},u_{m_k+1}),\frac{d(u_{n_k},u_{m_k+1})+d(u_{m_k},u_{n_k+1})}{2}\right\}\\ &\leq\max\left\{d(u_{n_k},u_{m_k}),d(u_{n_k},u_{n_k+1}),d(u_{m_k},u_{m_k+1}),\frac{d(u_{n_k},u_{m_k})+d(u_{m_k},u_{m_k+1})+d(u_{m_k},u_{n_k})+d(u_{n_k},u_{n_k+1})}{2}\right\}. \end{aligned}$$

It follows from (10) that

$$\lim_{k\to\infty} M(u_{n_k},u_{m_k},u_{n_k+1},u_{m_k+1})=\lim_{k\to\infty} d(u_{n_k},u_{m_k}). \tag{17}$$

Taking $k\to\infty$, it follows from (14), (16), (17) and the property of ψ that

$$1=\lim_{k\to\infty}\frac{\psi(d(u_{n_k+1},u_{m_k+1}))}{\psi(d(u_{n_k},u_{m_k}))}\leq\lim_{k\to\infty}\gamma(u_{n_k},u_{m_k})\leq 1.$$

By the property of γ and $\lim_{k\to\infty} d(u_{m_k}, u_{n_k}) = \varepsilon$,

$$0 = \lim_{n\to\infty} d(u_{n_k}, u_{m_k}) = \varepsilon > 0$$

which is impossible. Therefore, $\{u_n\}$ is a Cauchy sequence as claimed.

Now, $\{u_n\}$ is a Cauchy sequence in A_0 which is closed in *X*. Then, there exists $u \in A_0$ such that $\lim_{n\to\infty} u_n = u$. By (6), we have that

$$d(u_{n+1}, gx_{n+1}) = d(u_n, fx_{n+1}) = d(A,B).$$

Since (4) and (f,g) is *G*–proximally commutative edge preserving,

$$gu_n = fu_{n+1} \text{and} (u_n, u_{n+1}) \in E_G \text{ for all } n \geq 0.$$

It follows from the *G*–continuity of *f* and *g* that

$$gu = \lim_{n\to\infty} gu_n = \lim_{n\to\infty} fu_{n+1} = fu.$$

By Lemma 3.1, we finally have that $\mathcal{CB}(f,g) \neq \varnothing$.

For the uniqueness part, let y^* be another point in $\mathcal{CB}(f,g)$. Then, we have that $(fx^*, gx^*) \in E_G$, $(fy^*, gy^*) \in E_G$ and

$$d(x^*, fx^*) = d(y^*, fy^*) = d(A,B) = d(x^*, gx^*) = d(y^*, gy^*).$$

Here $M(x^*, y^*, x^*, y^*) = \max\{d(x^*, y^*), d(x^*, x^*), d(y^*, y^*)\} = d(x^*, y^*)$. Since (f,g) is *G*–proximally dominating,

$$\psi(d(x^*, y^*)) \leq \gamma(x^*, y^*)\psi(d(x^*, y^*)) \leq \psi(d(x^*, y^*)).$$

Since $d(x^*, y^*) > 0$, then $\psi(d(x^*, y^*)) > 0$. Thus $\gamma(x^*, y^*) = 1$. Lastly, by the property of γ,

$$d(x^*, y^*) = 0$$

The proof is now completed.

Theorem 3.3. *Suppose that all the hypotheses in Theorem 3.2 but the condition (iii) are satisfied. In stead of the condition (iii), we assume*

For a sequence $\{x_n\}$ in *A* satisfying $(fx_n, gx_{n+1}) \in E_G$ for all *n* and $x_n \to x \in A$, there exists a subsequence $\{x_{n_k}\}$ such that

$$d(A,B) = d(x, gx_{n_k}) = d(x, fx_{n_k}) \text{ for all } k.$$

Then, $\mathcal{CB}(f,g) \neq \varnothing$. Moreover, if $(fx, gx) \in E_G$ for all $x \in \mathcal{CB}(f,g)$, then (f,g) has a unique common best proximity point.

Proof. Following the proof of Theorem 3.2, there exists $u \in A_0$ such that $\lim_{n\to\infty} u_n = u$. By (5), we have that $d(u_n, gx_n) = d(u_{n-1}, fx_n) = d(A,B)$. Since (f,g) is *G*–proximally commutative edge preserving,

$$gu_{n-1} = fu_n$$

for all $n \geq 1$. Since $(fu_n, gu_{n+1}) = (fu_n, fu_n) \in E_G$, for all $n \in \mathbb{N}$, $\lim_{n\to\infty} u_n = u$. By (iii*), there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$d(A,B) = d(u, gu_{n_k}) = d(u, fu_{n_k}).$$

Since (f,g) is *G*–proximally commutative edge preserving, we obtain that $fu = gu$. Thus by Lemma 3.1, $\mathcal{CB}(f,g) \neq \varnothing$.

Example 3.4. Let $X = \mathbb{R}^2$ be equipped with the Euclidean metric d, and be endowed with the graph G where

$$E_G = \{((x,y),(u,v)) \in \mathbb{R}^2 \times \mathbb{R}^2; x \geq u \text{ and } y \leq v\}.$$

Consider the subsets $A = \{(x,2) : 0 \leq x \leq 7\}$ and $B = \{(x,-2) : 0 \leq x \leq 7\}$. It is clear that $d(A,B) = 4$, $A_0 = A$ and $B_0 = B$.

Define the mappings $f,g : A \to B$ by

$$f(x,2) = (x,-2) \text{ and } g(x,2) = (\ln(1+x),-2)$$

for all $(x,2) \in A$. Observe that f and g are continuous.

First, we will show that (f,g) is G–proximally commutative edge preserving.

Let $(f(x^*,2), f(y^*,2)) = ((x^*,-2),(y^*,-2)) \in E_G$. Then, $x^* \geq y^*$ and so $\ln(1+x^*) \geq \ln(1+y^*)$. Thus

$$(g(x^*,2), g(y^*,2)) = ((\ln(1+x^*),-2),(\ln(1+y^*),-2)) \in E_G.$$

Let $x,u,v \in A$ such that $(fx,gx) \in E_G$ and

$$d(u,gx) = d(A,B) = d(v,fx).$$

Note that $x = (\hat{x},2), u = (\hat{u},2), v = (\hat{v},2)$ where $\hat{u} = \ln(1+\hat{x}), \hat{v} = \hat{x}$ and $\hat{x} \geq \ln(1+\hat{x})$. This implies that $\hat{v} \geq \hat{u}$, and thus $(v,u) \in E_G$. Next, let $x,u,v \in A$ such that

$$d(u,gx) = d(v,fx) = d(A,B).$$

Then, $x = (\hat{x},2), u = (\hat{u},2), v = (\hat{v},2)$, where $\hat{u} = \ln(1+\hat{x})$ and $\hat{v} = \hat{x}$. Thus,

$$gv = (\ln(1+\hat{v}),-2) = (\ln(1+\hat{x}),-2) = (\hat{u},-2) = fu.$$

Thus, (f,g) is G–proximally commutative edge preserving.

To show that the pair (f,g) is G–proximally dominating, we choose the map $\gamma \in \Gamma$ given by

$$\gamma(x,y) = \begin{cases} 1, & \mathrm{d(x,y)} = 0, \\ \dfrac{\ln(1+d(x,y))}{d(x,y)}, & \mathrm{d(x,y)} > 0, \end{cases}$$

and $\psi \in \Psi$ given by $\psi(t) = 2t$ for all $t \geq 0$.

Let $x_1, x_2, u_1, u_2, v_1, v_2 \in A$ such that $(fx_1, gx_1), (fx_2, gx_2) \in E_G$ satisfying

$$d(u_1, gx_1) = d(u_2, gx_2) = d(A,B) = d(v_1, fx_1) = d(v_2, fx_2)$$

where $\hat{u}_1 = \ln(1+\hat{x}_1), \hat{u}_2 = \ln(1+\hat{x}_2), \hat{v}_1 = \hat{x}_1$, $\hat{v}_2 = \hat{x}_2$ and $\hat{x}_1, \hat{x}_2 \in [0,7]$.

To obtain the inequality (1), if $u_1 = u_2$ or $v_1 = v_2$, then we are done. We assume that $u_1 \neq u_2$. Then $\hat{u}_1, \hat{u}_2, \hat{v}_1, \hat{v}_2$ are all distinct, so $M(v_1, v_2, u_1, u_2) > 0$. Consider

$$\begin{aligned}
\psi(d(u_1,u_2)) &= 2d(u_1,u_2) = 2\,|\,\hat{u}_1 - \hat{u}_2\,| = 2\,|\ln(1+\hat{v}_1) - \ln(1+\hat{v}_2)\,| \\
&= 2\left|\ln\left(\frac{1+\hat{v}_2+\hat{v}_1-\hat{v}_2}{1+\hat{v}_2}\right)\right| \\
&\leq 2\ln\left(1+|\,\hat{v}_1 - \hat{v}_2\,|\right) = 2\ln\left(1+d(v_1,v_2)\right) \\
&\leq 2\ln\left(1+d(v_1,v_2)\right)\frac{M(v_1,v_2,u_1,u_2)}{d(v_1,v_2)} \\
&= \left[\frac{\ln\left(1+d(v_1,v_2)\right)}{d(v_1,v_2)}\right] 2M(v_1,v_2,u_1,u_2) \\
&= \gamma(v_1,v_2)\psi(M(v_1,v_2,u_1,u_2)).
\end{aligned}$$

Therefore, (f,g) is G–proximally dominating.

Notice also that

$$g(A_0)=\{(x,-2):0\le x\le \ln 8\}\subseteq\{(x,-2):0\le x\le 7\}=B_0=f(A_0).$$

Applying Theorem 3.2, there is a unique common best proximity point of the pair (f,g) which is, in fact, the point $(0,2)$.

4. Some Consequences

Let us proceed with further investigation on some particular circumstances where $\psi(t)=t$ and $\gamma(s,t)=k$ for $k\in[0,1)$, and $\psi(t)=t$ and $\gamma(s,t)=\dfrac{1}{1+(s+t)}$. The following are consequences of Theorem 3.2.

Corollary 4.1. *Suppose that the conditions (i) and (ii) from Theorem 3.2 hold. Moreover, suppose that the following are satisfied:*

1. if there exists $k\in[0,1)$ such that for each $x_1,x_2,u,v,x,y\in A$, $d(u,v)\le kM(x,y,u,v)$ whenever $(fx_1,gx_1)\in E_G$, $(fx_2,gx_2)\in E_G$ and $d(u,gx_1)=d(v,gx_2)=d(A,B)=d(x,fx_1)=d(y,fx_2)$;
2. either
   (a) f and g are G–continuous; or
   (b) for a sequence $\{x_n\}$ in A such that $(fx_n,gx_{n+1})\in E_G$ and $x_n\to x\in A$, there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(A,B)=d(x,gx_{n_k})=d(x,fx_{n_k}).$$

Then, $\mathcal{CB}(f,g)\ne\varnothing$.

Corollary 4.2. *Suppose that all the assumptions but the condition (C1) in Corollary 4.1 hold. Instead, it is replaced by*

For each $x_1,x_2,u,v,x,y\in A$, $d(u,v)\le\dfrac{M(x,y,u,v)}{1+M(x,y,u,v)}$, whenever $(fx_1,gx_1)\in E_G$, $(fx_2,gx_2)\in E_G$ and $d(u,gx_1)=d(v,gx_2)=d(A,B)=d(x,fx_1)=d(y,fx_2)$.

Then, $\mathcal{CB}(f,g)\ne\varnothing$.

5. Conclusion and Remarks

A certain class of contraction mappings has been presented, see Definition 2.5 in order to achieve common best proximity results. The main result is Theorem 3.2, which also gives rise to Theorem 3.3 by replacing one condition of the former. Following the main rerults, Corollaries 4.1 and 4.2 have been listed and served as particular cases. It is worth remarking that the above results may be established in the language of binary relations. To be more precise, one may regard a graph G as a binary relation on X, which analogously produces common best proximity point results in a metric space endowed with a binary relation.

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