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Some new applications of Integral and Differential operator for new subclasses of analytic functions

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Abstract

In this paper, we consider convolution operator

$$zD_{\varkappa}h(z) = h(z) * \frac{z}{(1-\varkappa z)(1-z)}$$

and define a new differential operator $D^m_{\lambda,\varkappa}$ on the analytic functions in the complex plane, for all $\varkappa$, $|\varkappa| \leq 1$. Furthermore, we consider this operator $D^m_{\lambda,\varkappa}$ and then define two new integral operators. We discuss some interesting applications of these operators by introducing several new subclasses of analytic functions. Also geometric properties are investigated for integral operators on new subclasses of analytic functions and some subordination results are discussed for differential operator $D^m_{\lambda,\varkappa}$.

Key words: Analytic functions, Fractional derivatives, Integral operators, Convolution, Starlike and Convex functions, Differential equations, Subordination, Mathematical operator.

Mathematics Subject Classification: Primary 05A30, 30C45, Secondary 11B65, 47B38

1. Introduction and Definitions

Let $\mathcal{H}(\nabla)$ represents the space of holomorphic functions in open unit disc $\nabla = \{z \in \mathbb{C} : |z| < 1\}$. Let

$$\mathcal{A}_n = \left\{h \in H(\nabla) : h(z) = z + a_{n+1}z^{n+1} + \ldots\right\}.$$

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For $n=1$, then $\mathcal{A}_1=\mathcal{A}$. The class $\mathcal{T}$ is a subclass of $\mathcal{A}$, and every $h\in\mathcal{T}$ has a series of the form:

$$\mathcal{T}=\left\{h\in\mathcal{A}:h(z)=z-{}_{j=2}^{\infty}\,a_jz^j,\,z\in\nabla\right\}.$$

Let the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α, $0\le\alpha<1$ is given by

$$\mathcal{S}^*(\alpha)=\left\{h\in\mathcal{A}:Re\left(\frac{zh'(z)}{h(z)}\right)>\alpha\right\}.$$

For $\alpha=0$, then $\mathcal{S}^*(\alpha)=\mathcal{S}^*$.

Let the class $\mathcal{C}(\alpha)$ of convex functions of order α, $0\le\alpha<1$ is given by

$$\mathcal{C}(\alpha)=\left\{h\in\mathcal{A}:Re\left(\frac{(zh'(z))'}{h'(z)}\right)>\alpha\right\}.$$

For $\alpha=0$, then $\mathcal{C}(\alpha)=\mathcal{C}$.

The convolution of the functions $h,\ g\in\mathcal{A}$, is define by

$$(h*g)(z)=z+{}_{j=2}^{\infty}\,a_jb_jz^j,\,z\in\nabla,$$

where

$$g(z)=z+{}_{j=2}^{\infty}\,b_jz$$

and

$$h(z)=z+{}_{j=2}^{\infty}\,a_jz. \tag{1.1}$$

Let the subordination [1] of two analytic functions h and g in ∇, (written as $h\prec g$), if there exist an analytic function w in ∇, along the following conditions

$$w(0)=0,\text{and}\left|w(z)\right|<1$$

for all $z\in\nabla$, such that

$$h(z)=g(w(z)).$$

If for any univalent function g, then

$$h\prec g$$

if and only if

$$h(0)=g(0)$$

and

$$h(\nabla)\subseteq g(\nabla).$$

Let $\psi:\mathbb{C}^3\times\nabla\to\mathbb{C}$, and φ be a univalent and p is analytic in ∇ and satisfies the second-order differential subordination

$$\psi(p(z),zp'(z),z^2p''(z);z)\prec\varphi(z),\,\text{for}\,z\in\nabla, \tag{1.2}$$

then, p is called a solution of the differential subordination (see [1]). If $p\prec r$ for all r fulfilling (1.2), then the univalent function r is said to be a dominant of the solutions of the differential subordination, or just a dominant. The best dominant of (1.2) is a dominant that fulfils $\bar{r}$ for all dominants $\bar{r}$. The best dominant is unique up to a rotation of ∇.

The area of study of differential and integral operators has got the thoughtful attention of researchers due to many applications in various branches of mathematics. The field of study connecting with differential and integral operators is the hot topic for the researchers in the field of study on analytic and univalent functions. In 1951, Alexander [2] was the first who introduced integral operator. After that, Libra introduced Libra integral operator [3] while in [4] Livingston invented differential operator and still different type of differential and integral operators are introducing by the researchers see [5–10]. The invention of Salagean and Ruscheweyh operators play significant role in the field of mathematics as well as in Geometric Function Theory (GFT). Many researchers [11–14] discussed their applications and used it to defined many new subclasses of analytic, meromorphic and harmonic functions. Recently, researchers are started to involve the quantum calculus to discuss many new applications of differential and integral operators and they are producing remarkable outcomes which can apply to other areas of mathematics and physics. In article [15], author points out interesting applications of operators. Jackson [16] was first who brought the q-extension of ordinary derivative and introduced the q-derivative and q-integral operator and numerous mathematicians have discussed their applications from different point of view. Recently, number of researchers have investigated extensions of previously known operators, like Govindaraj and Sivasubramanian [17] gave the q-extension of Salagean differential operator and Kanas and Raducanu [18] used the basic concepts of q-calculus and gave the q-extension of Ruschewey operator and q-extension of other operators are investigated in [19–24]. Since operators are use to the study of differential equations by means of operator theory and functional analysis. In [25], number of integral operators have been introduced connected with the Lommel functions of the first kind and also derived various interesting mapping and geometric properties for the integral operators. Particularly, these integral operators play a very useful role in the study of pure and applied mathematical sciences. On this line of investigation, we introduced two new integral operators in this paper by using the newly defined differential operator $D^m_{\lambda,\varkappa}$.

Let $q \in (0,1)$ and $\eta \in \mathbb{C}$, the q-number defined as:

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [0]_q = 0, x \in \mathbb{C},$$
$$[n]_q = 1 + q + q^2 + \ldots + q^{n-1}, x = n \in \mathbb{N}.$$

Jackson [16] defined the q-derivative operator (D_q) for analytic functions h as follows:

$$\begin{aligned} D_q h(z) &= \frac{h(z)-h(qz)}{(1-q)z}, \; z \neq 0, 0 < q < 1, \\ &= 1 + \sum_{j=2}^{\infty} [j]_q a_j z^{j-1}. \end{aligned} \tag{1.3}$$

Also Jackson [26] defined the q-integral for the function h as follows:

$$h(z) d_q z = (1-q) z \sum_{j=0}^{\infty} h(q^j z) q^j.$$

Argawal and Sahoo considered q-derivative in [27] and defined a new class $\mathcal{S}^*_q(\alpha)$. Let an analytic function $h \in \mathcal{S}^*_q(\alpha), 0 \leq \alpha < 1$, if

$$\left| \frac{z D_q h(z)}{h(z)} - \frac{1-\alpha q}{1-q} \right| \leq \frac{1-\alpha}{1-q}.$$

Remark 1.1: *If* $q \to 1-$, *then* $\mathcal{S}_q^*(\alpha) = \mathcal{S}^*(\alpha)$.

Remark 1.2: *If* $\alpha = 0$, *the class* $\mathcal{S}_q^*(\alpha) = \mathcal{S}_q^*$, *studied by Ismail et al. in [28].*

Here, for complex number $\varkappa$, $|\varkappa| \le 1$, we consider convolution operator

$$zD_\varkappa h(z) = h(z) * \frac{z}{(1-\varkappa z)(1-z)} \tag{1.4}$$

$$\begin{aligned} &= \sum_{j=1}^{\infty} a_j z^j * \sum_{j=1}^{\infty} \frac{1-\varkappa^j}{1-\varkappa} z^j \\ &= z + \sum_{j=2}^{\infty} \frac{1-\varkappa^j}{1-\varkappa} a_j z^j \\ &= z + \sum_{j=2}^{\infty} [j]_\varkappa a_j z^j, \end{aligned} \tag{1.5}$$

where

$$[j]_\varkappa = \frac{1-\varkappa^j}{1-\varkappa}.$$

Take $\varkappa = q$ in (1.5), then we have q-derivative $D_q h$ and is a special case of the convolution operator (1.4).

Now considering the operator $zD_\varkappa h(z)$, we define a new differential operator $D_{\lambda,\varkappa}^m$ on the analytic functions in the open unit disk ∇.

Definition 1.3: *For* $\lambda \ge 0, m \in \mathbb{N}$, $\varkappa \in \mathbb{C}$, $|\varkappa| \le 1$ *and* $h \in \mathcal{A}$, *the operator* $D_{\lambda,\varkappa}^m : \mathcal{A} \to \mathcal{A}$, *is defined by*

$$\begin{aligned} D_{\lambda,\varkappa}^0 h(z) &= h(z), \\ D_{\lambda,\varkappa}^1 h(z) &= (1-\lambda)h(z) + \lambda z D_\varkappa h(z) = D_{\lambda,\varkappa} h(z), \\ &\vdots \\ D_{\lambda,\varkappa}^m h(z) &= (1-\lambda) D_\varkappa^{m-1} h(z) + \lambda z D_\varkappa (D_\varkappa^{m-1} h(z)) \\ &= D_{\lambda,\varkappa}(D_{\lambda,\varkappa}^{m-1} h(z), z \in \nabla. \end{aligned} \tag{1.6}$$

After some simple calculation of (1.4), (1.1) and (1.6), then we have

$$D_{\lambda,\varkappa}^m h(z) = z + \sum_{j=2}^{\infty} (\lambda([j]_\varkappa - 1) + 1)^m a_j z^j.$$

If $h(z) = z - \sum_{j=2}^{\infty} a_j z^j \in \mathcal{T}$, then we have

$$D_{\lambda,\varkappa}^m h(z) = z - \sum_{j=2}^{\infty} (\lambda([j]_\varkappa - 1) + 1)^m a_j z^j.$$

The following identity can also be verified:

$$D_{\lambda,\varkappa}^{m+1} h(z) = \left(1 - \frac{[\lambda]_\varkappa}{\varkappa^\lambda}\right) D_{\lambda,\varkappa}^m h(z) + \left(\frac{[\lambda]_\varkappa}{\varkappa^\lambda}\right) z (D_{\lambda,\varkappa}^m h(z))'. \tag{1.7}$$

Remark 1.4: *If* $h(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathcal{A}$, *then we can also write*

$$D_{\lambda,\varkappa}^m h(z) = z + \sum_{j=2}^{\infty} (\lambda([j]_\varkappa - 1) + 1)^m a_j^2 z^j$$

and if $h(z) = z - \sum_{j=2}^{\infty} a_j z^j \in \mathcal{T}$, then

$$D_{\lambda,\varkappa}^m h(z) = z - \sum_{j=2}^{\infty} (\lambda([j]_\varkappa - 1) + 1)^m a_j^2 z^j.$$

Remark 1.5: *If $\varkappa = 1$, then we obtain Al-Oboudi-differential operator [29].*

Remark 1.6: *If $\lambda = 1$, and $\varkappa = q$, then we get Salagean q-differential operator [17].*

Remark 1.7: *If $\lambda = 1$, $\varkappa = 1$, then we have Salagean differential operator [30].*

In this paper, the new differential operator $D^m_{\lambda,\varkappa}$ is defined, and then using the operator $D^m_{\lambda,\varkappa}$ given in the Definition 1.3 two new integral operators are introduced. Considering these newly defined operators, some new classes of analytic functions are formulated. Related to the new integral operators two new lemmas are proved which will be used to prove our main results. Next, we provide some theorems including the conditions for functions in class $\mathcal{T}$ and show the connection between new operators and newly defined classes. In the last, we used the best dominants of certain differential subordinations, we proved subordination results, which are the extended forms of Theorem 2.3 and Theorem 2.4.

Using the operator $D^m_{\lambda,\varkappa}h(z)$, we now define two new integral operators.

Definition 1.8 *Let integral operators $F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}$ and $G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}$ for functions $h_i \in \mathcal{T}$, $\gamma_i \in \mathbb{R}$, $i \in \{1,2,3\ldots,l\}$, are define as follows:*

$$F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l} = \int_0^z \left(\frac{D^m_{\lambda,\varkappa}h_1(t)}{t}\right)^{\gamma_1} \cdots \left(\frac{D^m_{\lambda,\varkappa}h_l(t)}{t}\right)^{\gamma l} dt, \tag{1.8}$$

and

$$G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l} = \int_0^z \left(\left(\frac{D^m_{\lambda,\varkappa}h_1(t)}{t}\right)'\right)^{\gamma_1} \cdots \left(\left(\frac{D^m_{\lambda,\varkappa}h_l(t)}{t}\right)'\right)^{\gamma l} dt, \tag{1.9}$$

where, $\lambda \geq 0$, $|\varkappa| \leq 1$, $m \in \mathbb{N}$, $z \in \nabla$.

Remark 1.9 *For $\lambda = 0$, $m = 0$, $\varkappa = 1$, in (1.8) then we have integral operator investigated in [31] and for $\lambda = 1$, $m = 0$, $\varkappa \to 1$, in (1.9) then we have integral operator investigated in [32].*

Considering the differential operator $D^m_{\lambda,\varkappa}$ and the integral operators $F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}$ and $G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}$, we define some new classes of analytic functions $h \in \mathcal{T}$.

Definition 1.10: *The class $\mathcal{R}(\delta,\varkappa)$, $\delta > 1$, of the functions $h \in \mathcal{T}$ which satisfy the inequality*

$$Re\left(\frac{z(D^m_{\lambda,\varkappa}h(z))'}{D^m_{\lambda,\varkappa}h(z)}\right) < \delta, z \in \nabla.$$

Definition 1.11: *The class $\mathcal{C}(\delta,\varkappa)$, $\delta > 1$, of the functions $h \in \mathcal{T}$ which satisfy the inequality*

$$Re\left(1 + \frac{z(D^m_{\lambda,\varkappa}h(z))''}{(D^m_{\lambda,\varkappa}h(z))'}\right) < \delta, z \in \nabla.$$

Definition 1.12: *The class $\mathcal{RA}(\beta,\mu,\varkappa)$, $0 \leq \beta < 1$, $0 < \mu \leq 1$, of the functions $h \in \mathcal{T}$ which satisfy the inequality*

$$\mu\left|\beta\left(\frac{z(D^m_{\lambda,\varkappa}h(z))'}{D^m_{\lambda,\varkappa}h(z)}\right) - 1\right| > \left|\frac{z(D^m_{\lambda,\varkappa}h(z))'}{D^m_{\lambda,\varkappa}h(z)} - 1\right|, z \in \nabla.$$

Definition 1.13: *The class* $\mathcal{CA}(\beta,\mu,\varkappa), 0 \le \beta < 1, 0 < \mu \le 1,$ *of the functions* $h \in \mathcal{T}$ *which satisfy the inequality*

$$\mu\left|\beta\left(1+\frac{z(D^m_{\lambda,\varkappa}h(z))''}{(D^m_{\lambda,\varkappa}h(z)))'}\right)+1\right| > \left|\frac{z(D^m_{\lambda,\varkappa}h(z))''}{(D^m_{\lambda,\varkappa}h(z))'}\right|, z \in \nabla.$$

Definition 1.14: *The class* $\mathcal{LAF}(\lambda,\beta,\mu,\gamma_{1,}\gamma_{2,\ldots}\gamma_l,\varkappa)$, $\lambda \ge 0$, $\beta \ge 0$, *and* $-1 \le \mu \le 1$ *contains the functions* h_i $i \in \{1,2,\ldots l\}$ *which satisfy*

$$\left(\beta\left|\frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_{1,}\gamma_{2,\ldots}\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_{1,}\gamma_{2,\ldots}\gamma_l}(z)\right)'}\right|+\mu\right) \le Re\left(1+\frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_{1,}\gamma_{2,\ldots}\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_{1,}\gamma_{2,\ldots}\gamma_l}(z)\right)'}\right), z \in \nabla,$$

where, $F^{m,\varkappa}_{\lambda,\gamma_{1,}\gamma_{2,\ldots}\gamma_l}(z)$ is defined in (1.8).

Definition 1.15: *The class* $\mathcal{LAG}(\lambda,\beta,\mu,\gamma_{1,}\gamma_{2,\ldots}\gamma_l,\varkappa)$, $\lambda \ge 0$, $\beta \ge 0$, *and* $-1 \le \mu \le 1$ *contains of the functions* h_i, $i \in \{1,2,\ldots l\}$ *which satisfy the inequality*

$$\left(\beta\left|\frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_{1,}\gamma_{2,\ldots}\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_{1,}\gamma_{2,\ldots}\gamma_l}(z)\right)'}\right|+\mu\right) \le Re\left(1+\frac{z\left(G^{m,\varkappa}_{\lambda,\gamma_{1,}\gamma_{2,\ldots}\gamma_l}\right)''}{\left(G^{m,\varkappa}_{\lambda,\gamma_{1,}\gamma_{2,\ldots}\gamma_l}\right)'}\right), z \in \nabla,$$

where, $G^{m,\varkappa}_{\lambda,\gamma_{1,}\gamma_{2,\ldots}\gamma_l}(z)$ is defined in (1.9).

2. Lemmas

Here, we state some known lemmas and also prove two new lemmas that will help to prove our main results of this article.

Lemma 2.1: [1]. *Let* p *is univalent in* ∇ *and let* ϕ *be an analytic in domain containing* $p(\nabla)$. *If*

$$\frac{zp'(z)}{\phi(p(z))} \tag{2.1}$$

is starlike, then

$$z\psi'(z)\phi(\psi(z)) \prec zp'(z)\phi(p(z)), z \in \nabla, \tag{2.2}$$

then,

$$\psi(z) \prec p(z),$$

and $p(z)$ is the best dominant.

Lemma 2.2: [33]. *Let* s *and* p *are analytic in* ∇ *and* p *is convex univalent, for complex number* α,β *and* γ *and* $\gamma \ne 0$. *Further let*

$$Re\left[\frac{\alpha}{\gamma}+\frac{2\beta}{\gamma}p(z)+\left(1+\frac{zp''(z)}{p(z)}\right)\right] > 0.$$

If an analytic function $s(z) = 1 + c_1 z + ...$ in ∇ and satisfies

$$\alpha s(z) + \beta s^2(z) + \gamma z s'(z) \prec \alpha p(z) + \beta p^2(z) + \gamma z p'(z),$$

then,

$$s(z) \prec p(z).$$

Also $p(z)$ is the best dominant.

Theorem 2.3: [34]. *For* $0 < \beta \le 1$, *let* p *be convex univalent and*

$$Re\left[\frac{1-\beta}{\beta} + 2p(z) + \left(1 + \frac{zp''(z)}{p(z)}\right)\right] > 0.$$

If $h \in \mathcal{A}$ satisfies

$$\frac{zh'(z)}{h(z)} + \beta z^2 \frac{h''(z)}{h'(z)} \prec (1-\beta) p(z) + \beta p^2(z) + \gamma z p'(z),$$

then

$$\frac{zh'(z)}{h(z)} \prec p(z).$$

Also $p(z)$ is the best dominant.

Theorem 2.4: [34]. *Let* $\varphi(z) = \dfrac{zp'(z)}{p(z)}$ *is starlike univalent in* ∇ *and* p *is analytic in* ∇ *and* $p(0) = 1$. *If* $h \in \mathcal{A}$ *satisfies the following subordination*

$$\frac{(zh(z))''}{h'(z)} - 2\frac{zh'(z)}{h(z)} \prec \varphi(z)$$

then,

$$\frac{z^2 h'(z)}{h^2(z)} \prec p(z).$$

Also $p(z)$ is the best dominant.

Lemma 2.5: *For* $h_i(z) = z -_{j=2}^{\infty} a_{i,j} z^j$, $i \in \{1,2,...l)$, *then we have*

$$\frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,...\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,...\gamma_l}(z)\right)''} =_{i=1}^{l} \gamma_i \left(\frac{-_{j=2}^{\infty}([j]_\varkappa - 1)\{\lambda([j]_\varkappa - 1) + 1\}^m a_{i,j}^2 z^{j-1}}{1 -_{j=2}^{\infty} \{\lambda([j]_\varkappa - 1) + 1\}^m a_{i,j}^2 z^{j-1}}\right),$$

where, $F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,...\gamma_l}(z)$ is defined in (1.8).

Proof. For $h_i(z) = z -_{j=2}^{\infty} a_{i,j} z^j$, $i \in \{1,2,...l)$, then

$$(D^m_{\lambda,\varkappa} h_i(z))'' = 1 -_{j=2}^{\infty} [j]_\varkappa \{\lambda([j]_\varkappa - 1) + 1\}^m (a_{i,j})^2 z^{j-1}.$$

We obtain

$$\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)' = \left(\frac{D^m_{\lambda,\varkappa}h_1(z)}{z}\right)^{\gamma_1} \ldots \left(\frac{D^m_{\lambda,\varkappa}h_l(z)}{z}\right)^{\gamma_l},$$

so

$$\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)'' = E_1\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)' \frac{z}{D^m_{\lambda,\varkappa}h_1(z)} + \ldots + E_l\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)' \frac{z}{D^m_{\lambda,\varkappa}h_l(z)},$$

where

$$E_i = \gamma_i\left(\frac{z\left(D^m_{\lambda,\varkappa}h_i(z)\right)' - D^m_{\lambda,\varkappa}h_i(z)}{z^2}\right),$$

where, $i \in \{1,2,3,\ldots l\}$. We calculate the expression

$$\frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)'} = \mathop{}_{i=1}^{l} \gamma_i\left[\frac{z\left(D^m_{\lambda,\varkappa}h_i(z)\right)'}{D^m_{\lambda,\varkappa}h_i(z)} - 1\right].$$

We find

$$\begin{aligned}\frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)'} &= \mathop{}_{i=1}^{l} \gamma_i\left(\frac{z - \mathop{}_{j=2}^{\infty}[j]_\varkappa\left\{\lambda\left([j]_\varkappa - 1\right)+1\right\}^m a^2_{i,j}z^j}{z - \mathop{}_{j=2}^{\infty}\left\{\lambda\left([j]_\varkappa - 1\right)+1\right\}^m a^2_{i,j}z^j} - 1\right)\\ &= \mathop{}_{i=1}^{l} \gamma_i\left(\frac{-\mathop{}_{j=2}^{\infty}\left([j]_\varkappa - 1\right)\left\{\lambda\left([j]_\varkappa - 1\right)+1\right\}^m a^2_{i,j}z^j}{z - \mathop{}_{j=2}^{\infty}\left\{\lambda\left([j]_\varkappa - 1\right)+1\right\}^m a^2_{i,j}z^j}\right)\\ &= \mathop{}_{i=1}^{l} \gamma_i\left(\frac{-\mathop{}_{j=2}^{\infty}\left([j]_\varkappa - 1\right)\left\{\lambda\left([j]_\varkappa - 1\right)+1\right\}^m a^2_{i,j}z^{j-1}}{1 - \mathop{}_{j=2}^{\infty}\left\{\lambda\left([j]_\varkappa - 1\right)+1\right\}^m a^2_{i,j}z^{j-1}}\right).\end{aligned}$$

Lemma 2.6: *For* $h_i(z) = z - \mathop{}_{j=2}^{\infty} a_{i,j}z^j$, $i \in \{1,2,\ldots l)$, *then we have*

$$\frac{z\left(G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)''}{\left(G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)'} = -\mathop{}_{i=1}^{l}\gamma_i\left(\frac{\mathop{}_{j=2}^{\infty}[j]_\varkappa\left([j-1]_\varkappa\right)\left\{\lambda\left([j]_\varkappa - 1\right)+1\right\}^m a^2_{i,j}z^{j-1}}{1 - \mathop{}_{j=2}^{\infty}[j]_\varkappa\left\{\lambda\left([j]_\varkappa - 1\right)+1\right\}^m a^2_{i,j}z^{j-1}}\right),$$

where, $G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)$ is defined in (1.9).

Proof. For $h_i(z) = z - \mathop{}_{j=2}^{\infty} a_{i,j}z^j$, $i \in \{1,2,\ldots l)$, we obtain

$$\left(G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)' = \left((D^m_{\lambda,\varkappa}h_1(z))'\right)^{\gamma_1} \ldots \left((D^m_{\lambda,\varkappa}h_1(z))'\right)^{\gamma_l},$$

so

$$\left(G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)'' ={}_{i=1}^{l} \gamma_i \left(G^{m,n,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)' \frac{\left(D^m_{\lambda,\varkappa} h_i(z)\right)''}{\left(D^m_{\lambda,\varkappa} h_i(z)\right)'}.$$

We calculate the expression $\dfrac{z\left(G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)''}{\left(G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)'}$

$$\frac{z\left(G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)''}{\left(G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)'} ={}_{i=1}^{l} \gamma_i \left[\frac{z\left(D^m_{\lambda,\varkappa} h_i(z)\right)''}{\left(D^m_{\lambda,\varkappa} h_i(z)\right)'}\right].$$

We find

$$\frac{z\left(G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)''}{\left(G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)'} ={}_{i=1}^{l} \gamma_i \left(\frac{-{}_{j=2}^{\infty}[j]_\varkappa\left([j-1]_\varkappa\right)\left\{\lambda\left([j]_\varkappa-1\right)+1\right\}^m a_{i,j}^2 z^{j-1}}{1-{}_{j=2}^{\infty}[j]_\varkappa\left\{\lambda\left([j]_\varkappa-1\right)+1\right\}^m a_{i,j}^2 z^{j-1}}\right).$$

Hence

$$\frac{z\left(G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)''}{\left(G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)'} = -{}_{i=1}^{l}\gamma_i \left(\frac{{}_{j=2}^{\infty}[j]_\varkappa\left([j-1]_\varkappa\right)\left\{\lambda\left([j]_\varkappa-1\right)+1\right\}^m a_{i,j}^2 z^{j-1}}{1-{}_{j=2}^{\infty}[j]_\varkappa\left\{\lambda\left([j]_\varkappa-1\right)+1\right\}^m a_{i,j}^2 z^{j-1}}\right).$$

3. Main Results

Sufficient conditions for $h_i \in \mathcal{LAF}(\lambda,\beta,\mu,\gamma_1,\gamma_{2,\dots}\gamma_l,\varkappa)$ and $h_i \in \mathcal{LAG}(\lambda,\beta,\mu,\gamma_1,\gamma_{2,\dots}\gamma_l,\varkappa)$, where

$$\lambda \ge 0, \beta \ge 0, -1 \le \mu \le 1.$$

Theorem 3.1: *Let* $h_i \in \mathcal{T},\ i \in \{1,2,3\dots l\}$. *Then* $h_i \in \mathcal{LAF}\left(\lambda,\beta,\mu,\gamma_1,\gamma_{2,\dots}\gamma_l,\varkappa\right)$ *if and only if*

$${}_{i=1}^{l}\gamma_i\left(\beta+1\right)\left(\frac{{}_{j=2}^{\infty}\left([j]_\varkappa-1\right)\left\{\lambda\left([j]_\varkappa-1\right)+1\right\}^m a_{i,j}^2}{1-{}_{j=2}^{\infty}\left\{\lambda\left([j]_\varkappa-1\right)+1\right\}^m a_{i,j}^2}\right) \le 1-\mu, \tag{3.1}$$

where, $\beta \ge 0,\ -1 \le \mu \le 1$.

Proof. When (3.1) holds, we have to show that

$$\beta\left|\frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)'}\right| - Re\left(\frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)'}\right) \le 1-\mu,$$

where, $\beta \geq 0,\ -1 \leq \mu \leq 1.$

We have

$$\beta\left|\frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)'}\right| - Re\left(\frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)'}\right) \leq (\beta+1)\left|\frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)'}\right|.$$

Using the Lemma 2.5, we have

$$\left(\beta+1\right)\left|\frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)'}\right| = \left(\beta+1\right)\left|{}^{l}_{i=1}\gamma_i\left(\frac{-{}^{\infty}_{j=2}\left(\left[j\right]_\varkappa-1\right)\left\{\lambda\left(\left[j\right]_\varkappa-1\right)+1\right\}^m a^2_{i,j}z^{j-1}}{1-{}^{\infty}_{j=2}\left\{\lambda\left(\left[j\right]_\varkappa-1\right)+1\right\}^m a^2_{i,j}z^{j-1}}\right)\right|$$

$$\leq\left(\beta+1\right){}^{l}_{i=1}\gamma_i\left\{\frac{{}^{\infty}_{j=2}\left(\left[j\right]_\varkappa-1\right)\left\{\lambda\left(\left[j\right]_\varkappa-1\right)+1\right\}^m a^2_{i,j}\left|z^{j-1}\right|}{1-{}^{\infty}_{j=2}\left\{\lambda\left(\left[j\right]_\varkappa-1\right)+1\right\}^m a^2_{i,j}\left|z^{j-1}\right|}\right\},$$

$$\leq\left(\beta+1\right){}^{l}_{i=1}\gamma_i\left\{\frac{{}^{\infty}_{j=2}\left(\left[j\right]_\varkappa-1\right)\left\{\lambda\left(\left[j\right]_\varkappa-1\right)+1\right\}^m a^2_{i,j}}{1-{}^{\infty}_{j=2}\left\{\lambda\left(\left[j\right]_\varkappa-1\right)+1\right\}^m a^2_{i,j}}\right\}\leq 1-\mu.$$

So, we deduce

$$\beta\left|\frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)'}\right| - Re\left(\frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)'}\right) \leq 1-\mu,$$

or equivalently

$$Re\left(1+\frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)'}\right) \geq \beta\left|\frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_{2,\dots}\gamma_l}(z)\right)'}\right| + \mu.$$

Hence $h_i \in \mathcal{LAF}\left(\lambda,\beta,\mu,\gamma_1,\gamma_{2,\dots}\gamma_l,\varkappa\right)$.

Conversely, suppose $h_i \in \mathcal{LAF}\left(\lambda,\beta,\mu,\gamma_1,\gamma_{2,\dots}\gamma_l,\varkappa\right)$. From Lemma 2.5 and (3.1), we obtain

$$1-{}^{l}_{i=1}\gamma_i\left[\frac{{}^{\infty}_{j=2}\left(\left[j\right]_\varkappa-1\right)\left\{\lambda\left(\left[j\right]_\varkappa-1\right)+1\right\}^m a^2_{i,j}\left|z^{j-1}\right|}{1-{}^{\infty}_{j=2}\left\{\lambda\left(\left[j\right]_\varkappa-1\right)+1\right\}^m a^2_{i,j}\left|z^{j-1}\right|}\right]$$

$$\geq\beta\left|{}^{l}_{i=1}\gamma_i\left|\frac{{}^{\infty}_{j=2}\left(\left[j\right]_\varkappa-1\right)\left\{\lambda\left(\left[j\right]_\varkappa-1\right)+1\right\}^m a^2_{i,j}z^{j-1}}{1-{}^{\infty}_{j=2}\left\{\lambda\left(\left[j\right]_\varkappa-1\right)+1\right\}^m a^2_{i,j}z^{j-1}}\right|\right|+\mu$$

$$\geq\beta{}^{l}_{i=1}\gamma_i\left(\frac{{}^{\infty}_{j=2}\left(\left[j\right]_\varkappa-1\right)\left\{\lambda\left(\left[j\right]_\varkappa-1\right)+1\right\}^m a^2_{i,j}z^{j-1}}{1-{}^{\infty}_{j=2}\left\{\lambda\left(\left[j\right]_\varkappa-1\right)+1\right\}^m a^2_{i,j}z^{j-1}}\right)+\mu,$$

which is equivalent to

$$\sum_{i=1}^{l}\beta\gamma_i\left[\frac{\sum_{j=2}^{\infty}\left(\left[j\right]_{\varkappa}-1\right)\left\{\lambda\left(\left[j\right]_{\varkappa}-1\right)+1\right\}^{m}a_{i,j}^{2}z^{j-1}}{1-\sum_{j=2}^{\infty}\left\{\lambda\left(\left[j\right]_{\varkappa}-1\right)+1\right\}^{m}a_{i,j}^{2}z^{j-1}}\right]$$
$$+\sum_{i=1}^{l}\gamma_i\left[\frac{\sum_{j=2}^{\infty}\left(\left[j\right]_{\varkappa}-1\right)\left\{\lambda\left(\left[j\right]_{\varkappa}-1\right)+1\right\}^{m}a_{i,j}^{2}z^{j-1}}{1-\sum_{j=2}^{\infty}\left\{\lambda\left(\left[j\right]_{\varkappa}-1\right)+1\right\}^{m}a_{i,j}^{2}z^{j-1}}\right]$$
$$\leq 1-\mu,$$

which reduces to

$$\sum_{i=1}^{l}(\beta+1)\gamma_i\left[\frac{\sum_{j=2}^{\infty}\left(\left[j\right]_{\varkappa}-1\right)\left\{\lambda\left(\left[j\right]_{\varkappa}-1\right)+1\right\}^{m}a_{i,j}^{2}z^{j-1}}{1-\sum_{j=2}^{\infty}\left\{\lambda\left(\left[j\right]_{\varkappa}-1\right)+1\right\}^{m}a_{i,j}^{2}z^{j-1}}\right]\leq 1-\mu,$$

when $z\to 1-$ along the real axis, we deduce the inequality (3.1).

If we take $\chi=q$ $(0<q<1)$, then we obtain a new result, which is given below in the form of a corollary.

Corollary 3.2 *Let* $h_i\in\mathcal{T}$, $i\in\{1,2,3...l\}$. *Then* $h_i\in\mathcal{LAF}\left(\lambda,\beta,\mu,\gamma_{1,}\gamma_{2,...}\gamma_l,q\right)$ *if and only if*

$$\sum_{i=1}^{l}\gamma_i\left(\beta+1\right)\left(\frac{\sum_{j=2}^{\infty}\left(\left[j\right]_{\varkappa}-1\right)\left\{\lambda\left(\left[j\right]_{q}-1\right)+1\right\}^{m}a_{i,j}^{2}}{1-\sum_{j=2}^{\infty}\left\{\lambda\left(\left[j\right]_{q}-1\right)+1\right\}^{m}a_{i,j}^{2}}\right)\leq 1-\mu,$$

where, $\beta\geq 0$, $-1\leq\mu\leq 1$.

Using the technique of the proof of Theorem 3.1, and Lemma 2.6, we obtain the Theorem 3.3.

Theorem 3.3: *Let* $h_i\in\mathcal{T}$, $i\in\{1,2,3...l\}$. *Then* $h_i\in\mathcal{LAG}\left(\lambda,\beta,\mu,\gamma_{1,}\gamma_{2,...}\gamma_l,\varkappa\right)$ *if and only if*

$$\sum_{i=1}^{l}\gamma_i\left(\beta+1\right)\left(\frac{\sum_{j=2}^{\infty}\left[j\right]_{\varkappa}\left(\left[j-1\right]_{\varkappa}\right)\left\{\lambda\left(\left[j\right]_{\varkappa}-1\right)+1\right\}^{m}a_{i,j}^{2}}{1-\sum_{j=2}^{\infty}\left[j\right]_{\varkappa}\left\{\lambda\left(\left[j\right]_{\varkappa}-1\right)+1\right\}^{m}a_{i,j}^{2}}\right)\leq 1-\mu,$$

where, $\beta\geq 0$, $-1\leq\mu\leq 1$.

Now we prove some properties of the integral operators $F_{\lambda,\gamma_1,\gamma_{2,...}\gamma_l}^{m,\varkappa}(z)$ and $G_{\lambda,\gamma_1,\gamma_{2,...}\gamma_l}^{m,\varkappa}(z)$ on the classes $\mathcal{R}\left(\delta,\varkappa\right)$, $\mathcal{C}\left(\delta,\varkappa\right)$, $\mathcal{RA}\left(\beta,\mu,\varkappa\right)$, and $\mathcal{CA}\left(\beta,\mu,\varkappa\right)$.

Theorem 3.4 *Let* $\gamma_i\in\mathbb{R}$, $\gamma_i>0, i\in\{1,2,3...l\}$, $h_i\in\mathcal{T}$ *and* $\left|\frac{\left(D_{\lambda,\varkappa}^{m}h_i(z)\right)'}{D_{\lambda,\varkappa}^{m}h_i(z)}\right|<M_i$, *If* $h_i\in\mathcal{RA}\left(\beta_i,\mu_i,\varkappa\right)$, *then* $F_{\lambda,\gamma_1,\gamma_{2,...}\gamma_l}^{m,\varkappa}(z)\in\mathcal{D}(\delta')$, *where*

$$\delta'=1+\sum_{i=1}^{l}\gamma_i\mu_i(\beta_iM_i+1), z\in\nabla.$$

Proof. It is clear from (1.8) that $F_{\lambda,\gamma_1,\gamma_{2,...}\gamma_l}^{m,\varkappa}\in\mathcal{T}$. On differentiating $F_{\lambda,\gamma_1,\gamma_{2,...}\gamma_l}^{m,\varkappa}(z)$ given by (1.8), we get

$$\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)' =_{i=1}^{l} \left(\frac{D^m_{\lambda,\varkappa}h_i(z)}{z}\right)^{\gamma_i}. \tag{3.2}$$

Taking the logarithm derivative of (3.2) and multiplying by z, we obtain

$$\frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)'} =_{i=1}^{l} \gamma_i \left[\frac{z\left(D^m_{\lambda,\varkappa}h_i(z)\right)'}{D^m_{\lambda,\varkappa}h_i(z)} - 1\right],$$

equivalently

$$1 + \frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)'} = 1 +_{i=1}^{l} \gamma_i \left[\frac{z\left(D^m_{\lambda,\varkappa}h_i(z)\right)'}{D^m_{\lambda,\varkappa}h_i(z)} - 1\right]. \tag{3.3}$$

Now we take a real part of both sides of (3.3), we have

$$Re\left(1 + \frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)'}\right) = 1 +_{i=1}^{l} \gamma_i \left[Re\left(\frac{z\left(D^m_{\lambda,\varkappa}h_i(z)\right)'}{D^m_{\lambda,\varkappa}h_i(z)}\right) - 1\right]$$

$$\leq 1 +_{i=1}^{l} \gamma_i \left|\frac{z\left(D^m_{\lambda,\varkappa}h_i(z)\right)'}{D^m_{\lambda,\varkappa}h_i(z)} - 1\right|.$$

Since $h_i \in \mathcal{RA}\left(\beta_i, \mu_i, \varkappa\right)$, we deduce that

$$Re\left(1 + \frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z)\right)'}\right) < 1 +_{i=1}^{l} \gamma_i\mu_i \left|\beta_i \frac{z\left(D^m_{\lambda,\varkappa}h_i(z)\right)'}{D^m_{\lambda,\varkappa}h_i(z)} + 1\right|$$

$$< 1 +_{i=1}^{l} \gamma_i\mu_i\beta_i \left|\frac{z\left(D^m_{\lambda,\varkappa}h_i(z)\right)'}{D^m_{\lambda,\varkappa}h_i(z)}\right| +_{i=1}^{l} \gamma_i\mu_i\beta_i$$

$$< 1 +_{i=1}^{l} \gamma_i\mu_i(\beta_i M_i + 1).$$

As

$$_{i=1}^{l}\gamma_i\mu_i(\beta_i M_i + 1) > 0, F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\ldots\gamma_l}(z) \in \mathcal{D}(\delta^{'}),$$

where

$$\delta' = 1 +_{i=1}^{l} \gamma_i\mu_i(\beta_i M_i + 1), z \in \nabla.$$

Let $l = 1$, $\gamma_1 = \gamma$, $\delta_1 = \delta$ and $h_1 = h$ in Theorem 3.4, we obtain the following corollary.

Corollary 3.5: *Let* $h \in \mathcal{T}$ *and* $\left|\frac{h'(z)}{h(z)}\right| < M$, *If* $h \in \mathcal{RA}(\beta, \mu, \varkappa)$, *then*

$$_0^z \left(\frac{h(t)}{t}\right)^{\gamma} d(t) \in \mathcal{D}(\delta'),$$

where

$$\delta' = 1 + \gamma\mu(\beta M + 1), \gamma \in \mathbb{R}, \gamma > 0, z \in \nabla.$$

Theorem 3.6: *Let* $\gamma_i > 0,\ \delta_i > 1,\ \gamma_i > 0, i \in \{1,2,3...l\},\ h_i \in \mathcal{T}$. *Then* $F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,...\gamma_l}(z) \in \mathcal{D}(\delta')$, *where*

$$\delta' = 1 +_{i=1}^{l} \gamma_i(\delta_i - 1), z \in \nabla.$$

Proof. From (3.3), we have

$$Re\left(1 + \frac{z\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,...\gamma_l}(z)\right)''}{\left(F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,...\gamma_l}(z)\right)'}\right) = 1 +_{i=1}^{l} \gamma_i Re\left(\frac{z\left(D^m_{\lambda,\varkappa}h_i(z)\right)'}{D^m_{\lambda,\varkappa}h_i(z)}\right) -_{i=1}^{l} \gamma_i$$

$$< 1 +_{i=1}^{l} \gamma_i\delta_i -_{i=1}^{l} \gamma_i = 1 +_{i=1}^{l} \gamma_i(\delta_i - 1).$$

Since $\delta_i > 1$, evidently $_{i=1}^{l}\gamma_i(\delta_i - 1) > 0$ and hence $F^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,...\gamma_l}(z) \in \mathcal{D}(\delta')$, with

$$\delta' = 1 +_{i=1}^{l} \gamma_i(\delta_i - 1), z \in \nabla.$$

Letting $l = 1,\ \gamma_1 = \gamma,\ \delta_1 = \delta$ and $h_1 = h$ in Theorem 3.6, we obtain the following corollary.

Corollary 3.7: *Let* $\gamma > 0,\ h \in \mathcal{R}(\delta)$ *with* $\delta > 1$. *Then* $_0^z\left(\frac{h(t)}{t}\right)^{\gamma} d(t) \in \mathcal{D}(\delta')$, *where*

$$\delta' = 1 + \gamma(\delta - 1), z \in \nabla.$$

Theorem 3.8: *Let* $\gamma_i > 0$, *and* $h_i \in \mathcal{D}(\delta_i)$, *for* $i \in \{1,2,3,...,l\}$ *with* $\delta_i > 1$. *Then* $G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,...\gamma_l}(z) \in \mathcal{D}(\delta')$, *where*

$$\delta' = 1 +_{i=1}^{l} \gamma_i(\delta_i - 1), z \in \nabla.$$

Proof. From (3.3), we have

$$Re\left(1 + \frac{z\left(G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,...\gamma_l}(z)\right)''}{\left(G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,...\gamma_l}(z)\right)'}\right) = 1 +_{i=1}^{l} \gamma_i Re\left(\frac{z\left(D^m_{\lambda,\varkappa}h_i(z)\right)''}{\left(D^m_{\lambda,\varkappa}h_i(z)\right)'}\right) -_{i=1}^{l} \gamma_i$$

$$< 1 +_{i=1}^{l} \gamma_i\delta_i -_{i=1}^{l} \gamma_i$$

$$= 1 +_{i=1}^{l} \gamma_i(\delta_i - 1).$$

Since $\delta_i > 1$, evidently $_{i=1}^{l}\gamma_i(\delta_i - 1) > 0$ and hence $G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,...\gamma_l}(z) \in \mathcal{D}(\delta')$, with

$$\delta' = 1 +_{i=1}^{l} \gamma_i(\delta_i - 1), z \in \nabla.$$

Letting $l = 1,\ \gamma_1 = \gamma,\ \delta_1 = \delta$ and $h_1 = h$ in Theorem 3.8, we obtain the corollary.

Corollary 3.9 *Let* $\gamma > 0,\ h \in \mathcal{D}(\delta)$ *with* $\delta > 1$. *Then* ${}_0^{z}(h'(t))^{\gamma}\, d(t) \in \mathcal{D}(\delta')$, *where*

$$\delta' = 1 + \gamma(\delta - 1), z \in \nabla.$$

Theorem 3.10: *Let* $\gamma_i \in \mathbb{R},\ \gamma_i > 0, i \in \{1,2,3...l\},\ h_i \in \mathcal{DA}\left(\beta_i, \mu_i, \varkappa\right)$ *and* $\left|\frac{(D^m_{\lambda,\varkappa} h_i(z))''}{(D^m_{\lambda,\varkappa} h_i(z))'}\right| < M_i$, *then* $G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\dots\gamma_l}(z) \in \mathcal{D}(\delta')$, *where*

$$\delta' = 1 +_{i=1}^{l} \gamma_i \mu_i(\beta_i M_i + 1), z \in \nabla.$$

Proof. From the definition of $G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\dots\gamma_l}$ given by (1.9), we have

$$Re\left(1 + \frac{z\left(G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\dots\gamma_l}(z)\right)''}{\left(G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\dots\gamma_l}(z)\right)'}\right) \leq_{i=1}^{l} \gamma_i \left|\frac{z\left(D^m_{\lambda,\varkappa} h_i(z)\right)''}{\left(D^m_{\lambda,\varkappa} h_i(z)\right)'}\right|$$

$$<_{i=1}^{l} \gamma_i \mu_i \left|\beta_i\left(1 + \frac{z\left(D^m_{\lambda,\varkappa} h_i(z)\right)''}{\left(D^m_{\lambda,\varkappa} h_i(z)\right)'}\right) + 1\right| + 1$$

$$< 1 +_{i=1}^{l} \gamma_i \mu_i \beta_i \left(1 + \left|\frac{z\left(D^m_{\lambda,\varkappa} h_i(z)\right)''}{\left(D^m_{\lambda,\varkappa} h_i(z)\right)'}\right|\right) +_{i=1}^{l} \gamma_i \mu_i + 1$$

$$< 1 +_{i=1}^{l} \gamma_i \mu_i \left[\beta_i(1 + M_i) + 1\right].$$

As

$${}_{i=1}^{l} \gamma_i \mu_i \left[\beta_i(1 + M_i) + 1\right] > 0,$$

we conclude that

$$G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\dots\gamma_l}(z) \in \mathcal{D}(\delta'),$$

where

$$\delta' = 1 +_{i=1}^{l} \gamma_i \mu_i [\beta_i(1 + M_i) + 1], z \in \nabla.$$

If we take $\chi = q\ (0 < q < 1)$, then we obtain a new result, which is given below in the form of a corollary.

Corollary 3.11: *Let* $\gamma_i \in \mathbb{R},\ \gamma_i > 0, i \in \{1,2,3...l\},\ h_i \in \mathcal{DA}\left(\beta_i, \mu_i, q\right)$ *and* $\left|\frac{(D^m_{\lambda,q} h_i(z))''}{(D^m_{\lambda,q} h_i(z))'}\right| < M_i$, *then* $G^{m,\varkappa}_{\lambda,\gamma_1,\gamma_2,\dots\gamma_l}(z) \in \mathcal{D}(\delta')$, *where*

$$\delta' = 1 +_{i=1}^{l} \gamma_i \mu_i(\beta_i M_i + 1), z \in \nabla.$$

Letting $l = 1,\ \gamma_1 = \gamma,\ M_1 = 1$ and $h_1 = h$ in Theorem 3.10, we have following corollary.

Corollary 3.12: *Let* $\gamma \in \mathbb{R}$, $\gamma > 0$, $h \in \mathcal{DA}(\beta, \mu, \varkappa)$ *and* $\left|\frac{h''(z)}{h'(z)}\right| < M$, *where* M *is fixed, then* ${}_0^z(h'(t))^\gamma d(t) \in \mathcal{D}(\delta')$, *where*

$$\delta' = 1 + \gamma\mu\beta(1 + M) + 1], z \in \nabla.$$

3.1 Subordination Results

Theorem 3.13: *Let* φ *be convex and univalent,* $\gamma \neq 0$ *and*

$$Re\left\{\frac{(1-\gamma)\varkappa^\lambda}{[\lambda]_\varkappa \gamma} + \frac{2\varkappa^\lambda}{[\lambda]_\varkappa}\varphi(z) + \left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right)\right\} > 0.$$

If $h \in \mathcal{T}$ satisfies the differential subordination

$$\frac{D_{\lambda,\varkappa}^{m+1}h(z)}{D_{\lambda,\varkappa}^{m}h(z)}\left(\gamma\left(\frac{D_{\lambda,\varkappa}^{m+2}h(z)}{D_{\lambda,\varkappa}^{m+1}h(z)}\right) + 1 - \gamma\right) \prec (1-\gamma)\varphi(z) + \gamma\varphi^2(z) + \frac{\gamma[\lambda]_\varkappa}{\varkappa^\lambda} z\varphi'(z), \tag{3.4}$$

then

$$\frac{D_{\lambda,\varkappa}^{m+1}h(z)}{D_{\lambda,\varkappa}^{m}h(z)} \prec \varphi(z), z \in \nabla,$$

and φ is best dominant.

Proof. Consider

$$p(z) = \frac{D_{\lambda,\varkappa}^{m+1}h(z)}{D_{\lambda,\varkappa}^{m}h(z)}, \tag{3.5}$$

we obtain

$$\frac{p'(z)}{p(z)} = \frac{D_{\lambda,\varkappa}^{m}h(z)}{D_{\lambda,\varkappa}^{m+1}h(z)}\left\{\frac{(D_{\lambda,\varkappa}^{m}h(z))(D_{\lambda,\varkappa}^{m+1}h(z))' - D_{\lambda,\varkappa}^{m+1}h(z)(D_{\lambda,\varkappa}^{m}h(z))'}{(D_{\lambda,\varkappa}^{m}h(z))^2}\right\} = \frac{(D_{\lambda,\varkappa}^{m+1}h(z))'}{D_{\lambda,\varkappa}^{m+1}h(z)} - \frac{(D_{\lambda,\varkappa}^{m}h(z))'}{D_{\lambda,\varkappa}^{m}h(z)}.$$

Therefore

$$\frac{zp'(z)}{p(z)} = \frac{z(D_{\lambda,\varkappa}^{m+1}h(z))'}{D_{\lambda,\varkappa}^{m+1}h(z)} - \frac{z(D_{\lambda,\varkappa}^{m}h(z))'}{D_{\lambda,\varkappa}^{m}h(z)}. \tag{3.6}$$

Using (1.7) in (3.6), we have

$$\frac{zp'(z)}{p(z)} = \frac{\varkappa^\lambda}{[\lambda]_\varkappa}\left(\frac{D_{\lambda,\varkappa}^{m+2}h(z)}{D_{\lambda,\varkappa}^{m+1}h(z)}\right) - \frac{\varkappa^\lambda}{[\lambda]_\varkappa}\left(1 - \frac{[\lambda]_\varkappa}{\varkappa^\lambda}\right) - \frac{\varkappa^\lambda}{[\lambda]_\varkappa}\left(\frac{D_{\lambda,\varkappa}^{m+1}h(z)}{D_{\lambda,\varkappa}^{m}h(z)}\right) + \frac{\varkappa^\lambda}{[\lambda]_\varkappa}\left(1 - \frac{[\lambda]_\varkappa}{\varkappa^\lambda}\right).$$

We obtain

$$\frac{[\lambda]_\varkappa}{\varkappa^\lambda}\left(\frac{zp'(z)}{p(z)}\right) = \left(\frac{D_{\lambda,\varkappa}^{m+2}h(z)}{D_{\lambda,\varkappa}^{m+1}h(z)}\right) - p(z).$$

So

$$\frac{D_{\lambda,\varkappa}^{m+2}h(z)}{D_{\lambda,\varkappa}^{m+1}h(z)}=\frac{[\lambda]_{\varkappa}}{\varkappa^{\lambda}}\left(\frac{zp'(z)}{p(z)}+\frac{\varkappa^{\lambda}}{[\lambda]_{\varkappa}}p(z)\right).$$

We deduce

$$\frac{D_{\lambda,\varkappa}^{m+1}h(z)}{D_{\lambda,\varkappa}^{m}h(z)}\left\{\gamma\frac{D_{\lambda,\varkappa}^{m+2}h(z)}{D_{\lambda,\varkappa}^{m+1}h(z)}+1-\gamma\right\}=p(z)\left\{\frac{\gamma[\lambda]_{\varkappa}}{\varkappa^{\lambda}}\left(\frac{zp'(z)}{p(z)}+\frac{\varkappa^{\lambda}}{[\lambda]_{\varkappa}}p(z)\right)+1-\gamma\right\}$$
$$=(1-\gamma)p(z)+\gamma p^{2}(z)+\frac{\gamma[\lambda]_{\varkappa}}{\varkappa^{\lambda}}zp'(z).$$

Hence, the differential subordination (3.4) becomes

$$(1-\gamma)p(z)+\gamma p^{2}(z)+\frac{\gamma[\lambda]_{\varkappa}}{\varkappa^{\lambda}}zp'(z)\prec(1-\gamma)\varphi(z)+\gamma\varphi^{2}(z)+\frac{\gamma[\lambda]_{\varkappa}}{\varkappa^{\lambda}}zp'(z).$$

Using Lemma 2.3, we get

$$\frac{DR_{\lambda,\varkappa}^{m+1}h(z)}{D_{\lambda,\varkappa}^{m}h(z)}\prec\varphi(z).$$

Also φ is the best dominant.

If we take $\chi=q$ $(0<q<1)$, then we obtain a new result, which is given below in the form of a corollary.

Corollary 3.14: *Let φ be convex and univalent, $\gamma\neq 0$ and*

$$Re\left\{\frac{(1-\gamma)q^{\lambda}}{[\lambda]_{q}\gamma}+\frac{2q^{\lambda}}{[\lambda]_{q}}\varphi(z)+\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right)\right\}>0.$$

If $h\in\mathcal{T}$ satisfies the differential subordination

$$\frac{D_{\lambda,q}^{m+1}h(z)}{D_{\lambda,q}^{m}h(z)}\left(\gamma\left(\frac{D_{\lambda,q}^{m+2}h(z)}{D_{\lambda,q}^{m+1}h(z)}\right)+1-\gamma\right)\prec(1-\gamma)\varphi(z)+\gamma\varphi^{2}(z)+\frac{\gamma[\lambda]_{q}}{q^{\lambda}}z\varphi'(z),$$

then

$$\frac{D_{\lambda,q}^{m+1}h(z)}{D_{\lambda,q}^{m}h(z)}\prec\varphi(z),z\in\nabla,$$

where φ is the best dominant.

Theorem 3.15: *Let φ be univalent in ∇, and $\frac{z\varphi'(z)}{\varphi(z)}$ be univalent and starlike in ∇, $\varphi(0)\neq 0$, $\gamma\neq 0$ If $h\in\mathcal{T}$ satisfies the differential subordination*

$$\frac{D_{\lambda,\varkappa}^{m+2}h(z)}{D_{\lambda,\varkappa}^{m+1}h(z)}-\gamma\frac{D_{\lambda,\varkappa}^{m+1}h(z)}{D_{\lambda,\varkappa}^{m}h(z)}\prec\frac{[\lambda]_{\varkappa}}{\varkappa^{\lambda}}\frac{z\varphi'(z)}{\varphi(z)}+1-\gamma,z\in\nabla, \tag{3.7}$$

then

$$\frac{z^{\gamma-1}D_{\lambda,\varkappa}^{m+1}h(z)}{(D_{\lambda,\varkappa}^{m}h(z))^{\gamma}} \prec \varphi(z), z \in \nabla. \tag{3.8}$$

and φ is best dominant.

Proof. Let

$$p(z) = \frac{z^{\gamma-1}D_{\lambda,\varkappa}^{m+1}h(z)}{(D_{\lambda,\varkappa}^{m}h(z))^{\gamma}}. \tag{3.9}$$

Taking the derivative, we get

$$p'(z) = \left\{\frac{z^{\gamma-2}(\gamma-1)\left(D_{\lambda,\varkappa}^{m+1}h(z)\right) + z^{\gamma-1}\left(D_{\lambda,\varkappa}^{m+1}h(z)\right)'}{\left(D_{\lambda,\varkappa}^{m}h(z)\right)^{\gamma}}\right\} - \frac{\gamma z^{\gamma-1}\left(D_{\lambda,\varkappa}^{m+1}h(z)\right)\left(D_{\lambda,\varkappa}^{m}h(z)\right)'}{\left(D_{\lambda,\varkappa}^{m}h(z)\right)\left(D_{\lambda,\varkappa}^{m}h(z)\right)^{\gamma}}.$$

Therefore,

$$\begin{aligned}\frac{zp'(z)}{p(z)} &= \frac{\left(D_{\lambda,\varkappa}^{m}h(z)\right)^{\gamma}}{z^{\gamma-1}D_{\lambda,\varkappa}^{m+1}h(z)}\left\{\begin{array}{c}\dfrac{z^{\gamma-2}(\gamma-1)\left(D_{\lambda,\varkappa}^{m+1}h(z)\right) + z^{\gamma-1}\left(D_{\lambda,\varkappa}^{m+1}h(z)\right)'}{\left(D_{\lambda,\varkappa}^{m}h(z)\right)^{\gamma}} \\ -\dfrac{\gamma z^{\gamma-1}\left(D_{\lambda,\varkappa}^{m+1}h(z)\right)\left(D_{\lambda,\varkappa}^{m}h(z)\right)'}{\left(D_{\lambda,\varkappa}^{m}h(z)\right)\left(D_{\lambda,\varkappa}^{m}h(z)\right)^{\gamma}}\end{array}\right\}, \\ &= (\gamma-1) + \frac{z\left(D_{\lambda,\varkappa}^{m+1}h(z)\right)'}{D_{\lambda,\varkappa}^{m+1}h(z)} - \gamma\frac{z\left(D_{\lambda,\varkappa}^{m}h(z)\right)'}{D_{\lambda,\varkappa}^{m}h(z)}.\end{aligned}$$

Using the Identity (1.7), we have

$$\frac{zp'(z)}{p(z)} = \frac{\varkappa^{\lambda}}{[\lambda]_{\varkappa}}\left(\frac{D_{\lambda,\varkappa}^{m+2}h(z)}{D_{\lambda,\varkappa}^{m+1}h(z)}\right) - \gamma\frac{\varkappa^{\lambda}}{[\lambda]_{\varkappa}}\left(\frac{D_{\lambda,\varkappa}^{m+1}h(z)}{D_{\lambda,\varkappa}^{m}h(z)}\right) + \frac{\varkappa^{\lambda}(\gamma-1)}{[\lambda]_{\varkappa}},$$

which is equivalent to

$$\frac{D_{\lambda,\varkappa}^{m+2}h(z)}{D_{\lambda,\varkappa}^{m+1}h(z)} - \gamma\left(\frac{D_{\lambda,\varkappa}^{m+1}h(z)}{D_{\lambda,\varkappa}^{m}h(z)}\right) = \frac{[\lambda]_{\varkappa}}{\varkappa^{\lambda}}\frac{zp'(z)}{p(z)} + (1-\gamma).$$

By hypothesis (3.7), we have

$$\frac{zp'(z)}{p(z)} \prec \frac{z\varphi'(z)}{\varphi(z)}.$$

From Lemma 2.1 we obtain

$$\frac{z^{\gamma-1}D_{\lambda,\varkappa}^{m+1}h(z)}{(D_{\lambda,\varkappa}^{m}h(z))^{\gamma}} \prec \varphi(z)$$

and φ is the best dominant.

If we take $\chi = q$ $(0 < q < 1)$, then we obtain a new result, which is given below in the form of a corollary.

Corollary 3.16: *Let* φ *be univalent in* ∇, *and* $\frac{z\varphi'(z)}{\varphi(z)}$ *be univalent and starlike in* ∇, $\varphi(0) \neq 0$, $\gamma \neq 0$ *If* $h \in \mathcal{T}$ *satisfies the differential subordination*

$$\frac{D_{\lambda,q}^{m+2}h(z)}{D_{\lambda,q}^{m+1}h(z)} - \gamma\frac{D_{\lambda,q}^{m+1}h(z)}{D_{\lambda,q}^{m}h(z)} \prec \frac{[\lambda]_q}{q^{\lambda}}\frac{z\varphi'(z)}{\varphi(z)} + 1 - \gamma, z \in \nabla,$$

then

$$\frac{z^{\gamma-1}D_{\lambda,q}^{m+1}h(z)}{(D_{\lambda,q}^{m}h(z))^{\gamma}} \prec \varphi(z), z \in \nabla.$$

and φ is best dominant.

4. Conclusion

The integral and differential operator theory is a broad discipline that has applications in many branches of mathematics and physics, as well as in other fields, such as quantum group theory, analytic number theory, numerical analysis, special polynomials, fractional calculus, and other related theories. The special functions and orthogonal polynomials have played a important role in mathematics, engineering, physics, and other research disciplines in recent decades.

In this study by introducing the new operator $D_{\lambda,\varkappa}^{m}$, two new integral operators $F_{\lambda,\gamma_1,\gamma_{2,\ldots}\gamma_l}^{m,\varkappa}(z)$ and $G_{\lambda,\gamma_1,\gamma_{2,\ldots}\gamma_l}^{m,\varkappa}(z)$ are defined. Considering these operators, some new classes of functions are introduced and studied. The integral operators are also considered on newly defined classes $\mathcal{R}(\delta,\varkappa)$, $\mathcal{C}(\delta,\varkappa)$, $\mathcal{RA}(\beta,\mu,\varkappa)$, $\mathcal{CA}(\beta,\mu,\varkappa)$. Using the operator $D_{\lambda,\varkappa}^{m}$ some subordination results have investigated for class $\mathcal{T}$ and these subordination results are the extended form of Theorem 2.3 and Theorem 2.4. If we take $\varkappa = q$ in (1.4), (1.6), (1.8) and (1.9), then we can obtain q-analogous of differential and integral operators and then all the results that are investigated in this article can be easily translated into q-calculus theory.

With the help of differential and integrals operators, we can study differential equations from the perspectives of operator theory and functional analysis. The operator method is a technique for solving differential equations that makes use of the properties of differential operators. In the future, research might be done to see whether these operators can be employed to provide solutions to PDEs. These newly developed operators might be explored for potential applications in the physical sciences or other applied disciplines. Further, we can define these classes with the help of fractional derivatives and other mathematical operators.

For future work, interested readers can organized several interesting subclasses of analytic, univalent, bi-univalent, q-valent and meromorphic functions by using the newly introduced integral and differential operators of this paper. Also, the symmetry properties of this newly introduced operator can be studied in future research directions.

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