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Certain new subclass of close-to-convex harmonic functions defined by a third-order differential inequality

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Abstract

In this paper, we define a new class $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$ of normalized harmonic functions in the unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ which satisfy the following third-order differential inequality

$$Re\left(s'(z)+\lambda zs''(z)+\delta z^2 s'''(z)-\left(\frac{L-1}{2M}\right)\right)>|u'(z)+\lambda zu''(z)+\delta z^2 u'''(z)|,$$

where, $\lambda\geq\delta\geq 0$, $1\leq L<M\leq -1$ and $M\neq 0$. First of all, we prove one-to-one correspondence between class $\mathcal{R}_{\lambda}^{\delta}(L,M)$ of analytic functions and class $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$ of harmonic functions. Next, we prove that every function $f\in\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$ is closed-to-convex in open unit disk U. Furthermore, we investigate various properties of the this class $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$, such as coefficient bounds, growth estimates, sufficient coefficient condition. We establish that class $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$ is closed under convex combination and convolution. We involve Gaussian hypergeometric function to discuss some applications of newly defined class of harmonic functions and construct harmonic polynomials which belong to the considered class $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$. We explore some new and known lemmas to prove our main results.

Keywords: Harmonic functions, Univalent functions, Close-to-convex functions, Gaussian hypergeometric function

Mathematics Subject Classification: Primary 05A30, 30C45, Secondary 11B65, 47B38

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1. Introduction and Definitions

Let $\mathcal{H}$ denote the class of functions of the form $\xi = s + \overline{u}$ which are harmonic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by

$$\xi(0) = \xi_z(0) - 1 = 0.$$

Every function $\xi = s + \overline{u} \in \mathcal{H}$ can be written in the series of the form:

$$\xi(z) = z + \sum_{l=2}^{\infty} a_l z^l + \sum_{l=1}^{\infty} b_l z^l,$$

where

$$s(z) = z + \sum_{l=2}^{\infty} a_l z^l, \text{ and } u(z) = \sum_{l=1}^{\infty} b_l z^l, \tag{1.1}$$

where s and u are analytic functions.

The harmonic function ξ is locally univalent in U if and only if its Jacobian

$$J_{\xi}(z) = \left|\xi_z(z)\right|^2 - \left|\xi_{\bar{z}}(z)\right|^2$$

is non-zero in U, (see [1]) and harmonic function ξ is sense preserving in U if $\left|s'(z)\right| > \left|u'(z)\right|$, (see [2, 3]).

Let $\mathcal{A}$ denote the set of all analytic functions in U and normalized by

$$\xi(0) = \xi'(0) - 1 = 0.$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ which contains set of all univalent functions. The starlike functions ($\mathcal{S}^*$), convex functions (K), and close-to-convex functions (C) are the subclasses of class $\mathcal{S}$ and these classes map unit disk U onto starlike, convex, and close-to-convex domains. The set of all functions that are univalent, harmonic and sense-preserving in U are denoted $\mathcal{S}_{\mathcal{H}}$ and also we can write as:

$$\mathcal{H}^0 = \{\xi \in \mathcal{H} : \xi_{\bar{z}}(0) = 0\} \text{ and } \mathcal{S}_{\mathcal{H}}^0 = \{\xi \in \mathcal{S}_{\mathcal{H}} : \xi_{\bar{z}}(0) = 0\}.$$

The harmonic starlike functions ($\mathcal{S}_{\mathcal{H}}^{0,*}$), harmonic convex functions ($K_{\mathcal{H}}^0$), and harmonic close-to-convex functions ($C_{\mathcal{H}}^0$) are the subclasses of class $\mathcal{S}_{\mathcal{H}}^0$.

Note that

$$\mathcal{A} \subset \mathcal{H} \text{ and } \mathcal{S} \subset \mathcal{S}_{\mathcal{H}}^0,$$

and

$$\mathcal{S}^* \subset \mathcal{S}_{\mathcal{H}}^{0,*}, K \subset K_{\mathcal{H}}^0 \text{ and } C \subset C_{\mathcal{H}}^0.$$

If co-analytic part $u(z) = 0$, then $\mathcal{H}^0$ reduces to class of analytic functions $\mathcal{A}$ and $\mathcal{S}_{\mathcal{H}}^0$ reduces to the set of univalent functions $\mathcal{S}$. Similarly if $u(z) = 0$, then the well known subclasses $\mathcal{S}_{\mathcal{H}}^{0,*}, K_{\mathcal{H}}^0$ and $C_{\mathcal{H}}^0$ of harmonic functions reduces to subclasses $\mathcal{S}^*$, K and C of univalent function class $\mathcal{S}$.

Ponnusamy et al. [3] defined a class of harmonic function $\xi \in \mathcal{H}^0$ and satisfies the following condition

$$Re(\xi_z(z)) > |\xi_{\bar{z}}(z)|, z \in U.$$

Furthermore, Li and Ponnusamy [4, 5] discussed univalent criteria and convexity of the partial sums for this class. Nagpal and Ravichandran [6] investigated a class of harmonic functions $\mathcal{W}_{\mathcal{H}}^{0}$, which satisfy the condition

$$Re(s'(z)+zs''(z))>\left|u'(z)+zu''(z)\right|, z\in U.$$

The class $\mathcal{W}_{\mathcal{H}}^{0}$ is harmonic analogue of the class $\mathcal{W}$ defined in [7] for $\xi\in\mathcal{A}$ and satisfy the condition

$$Re(\xi'(z)+z\xi''(z))>0, z\in U.$$

In 2019, Ghosh and Vasudevarao [8] gave the generalizations of class $\mathcal{W}_{\mathcal{H}}^{0}$ and introduced a new class $\mathcal{W}_{\mathcal{H}}^{0}(\delta)$ of harmonic close-to-convex functions which satisfy the following condition

$$Re(s'(z)+\delta zs''(z))>\left|u'(z)+\delta zu''(z)\right|, z\in U.$$

They found some useful properties for this class such as, radius of convexity, coefficient bounds and growth estimates. After that, Rajbala and Prajapat [9] gave the generalizations of $\mathcal{W}_{\mathcal{H}}^{0}(\delta)$ and defined a new class $\mathcal{W}_{\mathcal{H}}^{0}(\delta,\lambda)$ of harmonic functions $\xi=s+\overline{u}\in\mathcal{H}^{0}$ which satisfy the second-order differential inequality:

$$Re(s'(z)+\delta zs''(z)-\lambda)>\left|u'(z)+\delta zu''(z)\right|, \delta\geq 0, 0\leq\lambda<1, z\in U.$$

For this class, they produced harmonic polynomials involving Gaussian hypergeometric function. Very recently, Yaşar and Yalçın [10] defined the class $\mathcal{R}_{\mathcal{H}}^{0}(\delta,\gamma)$ of harmonic functions which satisfy the third-order differential inequality:

$$Re(s'(z)+\delta zs''(z)+\gamma z^{2}s'''(z))>\left|u'(z)+\delta zu''(z)+\gamma z^{2}u'''(z)\right|, \delta\geq\gamma\geq 0.$$

They investigated that any function $\xi\in\mathcal{R}_{\mathcal{H}}^{0}(\delta,\gamma)$ is close-to-convex. Moreover, they constructed coefficient bounds, growth estimates, sufficient coefficient condition, and convolution properties. For more study (see [11, 12, 6, 13]).

Motivated by the work of Yaşar and Yalçın [10], Rajbala and Prajapat [9], we define a new classes of analytic and harmonic functions in U.

Definition 1.1: *Let $\mathcal{R}_{\lambda}^{\delta}(L,M)$ denote the class of functions $\xi\in\mathcal{A}$ and satisfy the condition*

$$Re(\xi'(z)+\lambda z\xi'(z)+\delta z^{2}\xi''(z))>\left(\frac{L-1}{2M}\right), \tag{1.2}$$

where, $\lambda\geq\delta\geq 0$, $1\leq L<M\leq -1$ and $M\neq 0$.

Definition 1.2: *Let $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$, denote the class of functions $\xi=s+\overline{u}\in\mathcal{H}^{0}$ which satisfy the following third order differential inequality:*

$$Re\left(s'(z)+\lambda zs''(z)+\delta z^{2}s'''(z)-\left(\frac{L-1}{2M}\right)\right)>\left|u'(z)+\lambda zu''(z)+\delta z^{2}u'''(z)\right|, \tag{1.3}$$

where, $\lambda\geq\delta\geq 0$, $1\leq L<M\leq -1$ and $M\neq 0$.

Remark 1.3: *The class $\mathcal{R}_{\lambda}^{\delta}(L,M)\subset\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$.*

Special cases:

1. $\mathcal{R}_{\mathcal{H}}^{0,1,0}(1,-1)=\mathcal{W}_{\mathcal{H}}^{0}$, studied by Nagpal and Ravichandran in [6].
2. $\mathcal{R}_{\mathcal{H}}^{0,\alpha,0}(1,-1)=\mathcal{W}_{\mathcal{H}}^{0}(\alpha)$, defined by Ghosh and Vasudevarao in [8].
3. $\mathcal{R}_{\mathcal{H}}^{0,\delta,0}(1-2\lambda,-1)=\mathcal{W}_{\mathcal{H}}^{0}(\delta,\lambda)$, introduced by Rajbala and Prajapat in [9].

Definition 1.4: *Let a nonnegative real numbers sequence* $\{q_l\}_{l=0}^{\infty}$ *is called convex null sequence, if* $t_l \to 0$ *as* $l \to \infty$, *and*

$$t_0 - t_1 \geq t_1 - t_2 \geq t_2 - t_3 \geq \ldots \geq t_{l-1} - t_l \geq \ldots \geq 0.$$

Definition 1.5: Goodloe [14] *defined the convolution (or Hardamard product) of a harmonic function* $\xi = s + \overline{u}$ *as follows:*

$$\xi * \phi = s * \phi + \overline{u * \phi},$$

where, ϕ is an analytic function.

Definition 1.6: [15]. *The subordination of two analytic functions* s_1 *and* s_2 *can be written as:*

$$s_1(z) \prec s_2(z).$$

If there exists $v \in \mathcal{A}$, along with the condition

$$v(0) = 0, \text{and} \left|v(z)\right| < 1$$

for all $z \in U$, such that

$$s_1(z) = s_2(v(z)).$$

Furthermore, if the s_2 is univalent in U, then we have

$$s_1(z) \prec s_2(z) \Leftrightarrow \{s_1(0) = s_2(0) \text{ and } s_1(U) \subset s_2(U)\}.$$

Definition 1.7: *The Gaussian hypergeometric function defined as:*

$$_2F_1(c,d,e;z) = F(c,d,e;z) = \sum_{l=0}^{\infty} \frac{(c)_l (d)_l}{(e)_l l!} z^l, \ \ (z \in U), \tag{1.4}$$

where, $c, d, e \in \mathbb{C}$, $e \neq 0, -1, -2, \ldots$, and Pochhammer symbol $(c)_l$ defined as:

$$(c)_l = c(c+1)(c+2)\ldots(c+l-1)$$

and

$$(c)_0 = 1.$$

If $Re(e-c-d) > 0$, then $F(c,d,e;z)$ is convergent in $|z| \leq 1$. For $z = 1$, we get well-known Gauss formula given in [16] and defined as follows:

$$F(c,d,e;1) = \frac{\Gamma(e)\Gamma(e-c-d)}{\Gamma(e-c)\Gamma(e-d)} < \infty. \tag{1.5}$$

In this paper, we define a new class $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$ of functions $\xi = s + \overline{u} \in \mathcal{H}^0$ which satisfy the third-order differential inequality. In section 2, we construct some new and known lemmas, which will be used to derive our main results. In section 3, first, we provide a one-to-one correspondence between the classes $\mathcal{R}_{\lambda}^{\delta}(L,M)$ and $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$ and then we obtain coefficient bounds, growth estimates, and sufficient condition for the class $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$. We also prove that this class is closed. In fourth section, we involve the Gaussian hypergeometric function and construct harmonic polynomials which belong to the considered class $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$.

2. Lemmas

The following lemmas will be used to prove our main results.

Lemma 2.1: [2]. *Let* $\mathcal{T}_\mu = s + \mu u$ *is close to convex, for each* $\mu(|\mu| = 1)$, *and* $|s'(0)| < |u'(0)|$. *Then* $\xi = s + \overline{u}$ *is close to convex in U. Where, s and u are analytic in U.*

Lemma 2.2: ([17, 18]). *Let* $\phi(z)$ *be analytic in U with* $C_l \neq 0$ *and defined by*

$$\phi(z) = C_l z^l + C_{l+1} z^{l+1} + \dots$$

and let $z_0 \neq 0,\ z_0 \in U$, such that

$$|\phi(z_0)| = \max_{|z| \leq |z_0|} |\phi(z)|$$

then, there is $n \in \mathbb{R},\ n \geq l \geq 1$, such that

$$\frac{z_0 \phi'(z_0)}{\phi(z_0)} = n$$

and

$$Re\left(1 + \frac{z_0 \phi''(z_0)}{\phi'(z_0)}\right) \geq n.$$

Lemma 2.3: *If* $\mathcal{T} \in \mathcal{R}_\lambda^\delta(L, M)$ *with* $\lambda \geq \delta \geq 0$ *and* $1 \leq L < M \leq -1,\ M \neq 0$. *Then*

$$Re(\mathcal{T}'(z)) > 0,$$

hence, $\mathcal{T}$ is close-to-convex in *U*.

Proof. Let $\mathcal{T} \in \mathcal{R}_\lambda^\delta(L, M)$ and

$$\mathcal{T}'(z) + \lambda z \mathcal{T}''(z) + \delta z^2 \mathcal{T}'''(z) - \left(\frac{L-1}{2M}\right) = \Psi(z).$$

Hence, we can observe that $Re(\Psi(z)) > 0$, for $z \in U$.

Let for analytic function ϕ in *U* with the condition

$$\phi(0) = 0$$

and

$$\mathcal{T}'(z) = \frac{1 + \phi(z)}{1 - \phi(z)}, \phi(z) \neq 1.$$

Now we have to show that $|\phi(z)| < 1, \forall\ z \in U$. For this we take

$$\Psi(z) = \mathcal{T}'(z) + \lambda z \mathcal{T}''(z) + \delta z^2 \mathcal{T}'''(z) - \left(\frac{L-1}{2M}\right).$$

After some simple calculation, we have

$$\Psi(z) = \frac{1 + \phi(z)}{1 - \phi(z)} + \frac{2\lambda z \phi'(z)}{(1 - \phi(z))^2} + 2\delta \left\{\frac{\phi''(z)(1 - \phi(z)) + 2(\phi'(z))^2}{(1 - \phi(z))^3}\right\} - \left(\frac{L-1}{2M}\right),$$

and

$$\Psi(z)=\frac{1+\phi(z)}{1-\phi(z)}+\frac{2\lambda z\phi'(z)}{(1-\phi(z))^2}+2\delta\left(\frac{z\phi'(z)}{(1-\phi(z))^2}\right)\frac{z\phi''(z)}{\phi'(z)}+4\delta\frac{(z\phi'(z))^2}{(1-\phi(z))^3}-\left(\frac{L-1}{2M}\right).$$

Since ϕ is analytic in U and $\phi(0)=0$, if there is $z_0\in U$, such that

$$\max_{|z|\le|z_0|}|\phi(z)|=|\phi(z_0)|=1.$$

Therefore, by using the Lemma 2.2, we have

$$\phi(z_0)=e^{i\theta},\text{ and } z_0\phi'(z_0)=n\phi(z_0)=ne^{i\theta},\quad (n\ge 1, 0<\theta<2\pi)$$

and

$$Re\left(\frac{z_0\phi''(z_0)}{\phi'(z_0)}\right)\ge n-1.$$

For the point $z_0\in U$, we obtain

$$\begin{aligned}Re\Psi(z_0)&=Re\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}+\frac{2\lambda ne^{i\theta}}{(1-e^{i\theta})^2}+\frac{2\delta ne^{i\theta}}{(1-e^{i\theta})^2}\left(\frac{z_0\phi''(z_0)}{\phi'(z_0)}\right)+\frac{4\delta(ne^{i\theta})^2}{(1-e^{i\theta})^3}-\left(\frac{L-1}{2M}\right)\right)\\&=\frac{-\lambda n}{1-\cos\theta}-\frac{\delta n}{(1-\cos\theta)}Re\left(\frac{z_0\phi''(z_0)}{\phi'(z_0)}\right)+\frac{\delta n^2}{(1-\cos\theta)}-\left(\frac{L-1}{2M}\right)\\&=\frac{-\lambda n}{1-\cos\theta}+\frac{\delta n}{(1-\cos\theta)}(1-n)+\frac{\delta n^2}{(1-\cos\theta)}-\left(\frac{L-1}{2M}\right)\\&\le-\left(\frac{(\lambda-\delta)n}{(1-\cos\theta)}+\left(\frac{L-1}{2M}\right)\right)<0.\end{aligned}$$

Which opposes our hypothesis. Hence, we have proved that there is no $z_0\in U$, such that

$$|\phi(z_0)|=1.$$

Therefore,

$$|\phi(z)|<1,\ \forall z\in U.$$

Therefore, we have

$$Re(\mathcal{T}'(z))>0.$$

Lemma 2.4: [19]. *If* $\{t_l\}_{l=0}^{\infty}$ *is a convex null sequence, then*

$$t(z)=\frac{t_0}{2}+\sum_{l=1}^{\infty}t_l z^l$$

is analytic and

$$Re(t(z))>0,\text{ in } U.$$

Lemma 2.5: [20]. *Let* p *be the analytic function and satisfy the condition* $p(0)=1$ *and* $Re(p(z))>1/2$ *in U. Then, the function* $p*\mathcal{T}$ *takes values in the convex hull of the image of U under calT. Where,* $\mathcal{T}$ *is an analytic function in U.*

Lemma 2.6: *Let* $\mathcal{T} \in \mathcal{R}_\lambda^\delta(L,M)$, *then*

$$Re\left(\frac{\mathcal{T}(z)}{z}\right) > \frac{1}{2}.$$

Proof. Let $\mathcal{T} \in \mathcal{R}_\lambda^\delta(L,M)$, and $\mathcal{T}(z) = z + \sum_{l=2}^{\infty} A_l z^l$, then

$$Re\left(1 + \sum_{l=2}^{\infty} l[1+(l-1)(\lambda+\delta(l-2))]A_l z^{l-1}\right) > \left(\frac{L-1}{2M}\right), (z \in U),$$

and it is equivalent to $Re(p(z)) > 1/2$ in U, where

$$p(z) = 1 + \frac{1}{2\left(1-\left(\frac{L-1}{2M}\right)\right)} \sum_{l=2}^{\infty} l[1+(l-1)(\lambda+\delta(l-2))]A_l z^{l-1}.$$

Let us consider a sequence $\{t_l\}_{l=0}^{\infty}$ given by

$$t_0 = 1 \text{ and } t_{l-1} = \frac{2\left(1-\left(\frac{L-1}{2M}\right)\right)}{l[1+(l-1)(\lambda+\delta(l-2))]}, \text{for } l \geq 2.$$

Hence the sequence $\{t_l\}_{l=0}^{\infty}$ is convex null sequence. Now by using the Lemma 2.4, we have

$$t(z) = 1 + \sum_{l=2}^{\infty} \frac{2\left(1-\left(\frac{L-1}{2M}\right)\right)}{l[1+(l-1)(\lambda+\delta(l-2))]} z^{l-1}$$

is analytic and $Re(t(z)) > \frac{1}{2}$ in U. Setting

$$\frac{\mathcal{T}(z)}{z} = p(z) * \left(1 + \sum_{l=2}^{\infty} \frac{2\left(1-\left(\frac{L-1}{2M}\right)\right)}{l[1+(l-1)(\lambda+\delta(l-2))]} z^{l-1}\right)$$

and by using the Lemma 2.5, we obtain

$$Re\left(\frac{\mathcal{T}(z)}{z}\right) > \frac{1}{2}, z \in U.$$

Lemma 2.7: *Let* $\mathcal{T}_i \in \mathcal{R}_\lambda^\delta(L,M)$, *for* $i = 1,2$, *then* $\mathcal{T}_1 * \mathcal{T}_2 \in \mathcal{R}_\lambda^\delta(L,M)$.

Proof. Let

$$\mathcal{T}_1(z) = z + \sum_{l=2}^{\infty} A_l z^l$$

and

$$\mathcal{T}_2(z) = z + \sum_{l=2}^{\infty} B_l z^l,$$

then Hardamard product of $\mathcal{T}_1(z)$ and $\mathcal{T}_2(z)$ are defined by

$$\mathcal{T}(z) = (\mathcal{T}_1 * \mathcal{T}_2)(z) = z + \sum_{l=2}^{\infty} A_l B_l z^l.$$

Since

$$\mathcal{T}'(z) = \mathcal{T}_1'(z) * \frac{\mathcal{T}_2(z)}{z}, z\mathcal{T}''(z) = z\mathcal{T}_1''(z) * \frac{\mathcal{T}_2(z)}{z}, z^2\mathcal{T}'''(z) = z^2\mathcal{T}_1'''(z) * \frac{\mathcal{T}_2(z)}{z},$$

then we yield

$$\frac{\mathcal{T}'(z) + \lambda z\mathcal{T}''(z) + \delta z^2\mathcal{T}'''(z) - \left(\frac{L-1}{2M}\right)}{1-\left(\frac{L-1}{2M}\right)} = \left(\frac{\mathcal{T}_1'(z) + \lambda z\mathcal{T}_1''(z) + \delta z^2\mathcal{T}_1'''(z) - \left(\frac{L-1}{2M}\right)}{1-\left(\frac{L-1}{2M}\right)}\right) * \frac{\mathcal{T}_2(z)}{z}. \tag{2.1}$$

Since $\mathcal{T}_1 \in \mathcal{R}_\lambda^\delta(L, M)$, then

$$Re\left\{\frac{\mathcal{T}_1'(z) + \lambda z\mathcal{T}_1''(z) + \delta z^2\mathcal{T}_1'''(z) - \left(\frac{L-1}{2M}\right)}{1-\left(\frac{L-1}{2M}\right)}\right\} > 0.$$

By using the Lemma 2.6, we have

$$Re\left\{\frac{\mathcal{T}(z)}{z}\right\} > \frac{1}{2}.$$

Using the Lemma 2.5 to (2.1), we get

$$Re\left\{\frac{\mathcal{T}'(z) + \lambda z\mathcal{T}''(z) + \delta z^2\mathcal{T}'''(z) - \left(\frac{L-1}{2M}\right)}{1-\left(\frac{L-1}{2M}\right)}\right\} > 0.$$

Thus

$$\mathcal{T} = \mathcal{T}_1 * \mathcal{T}_2 \in \mathcal{T}_\lambda^\delta(L, M).$$

Lemma 2.8: [21]. *Let* $c, d \in \mathbb{C}\setminus\{0\}$, $e > 0$. *Then*

i. For $c, d > 0$, $e > c + d + 1$,

$$\sum_{l=0}^{\infty} (l+1)\frac{(c)_l (d)_l}{(e)_l (1)_l} = \frac{\Gamma(e)\Gamma(e-c-d-1)}{\Gamma(e-c)\Gamma(e-d)}(cd + e - c - d - 1).$$

ii. For $c, d > 0$, $e > c + d + 2$,

$$\sum_{l=0}^{\infty} (l+1)^2 \frac{(c)_l (d)_l}{(e)_l (1)_l} = \frac{\Gamma(e)\Gamma(e-c-d)}{\Gamma(e-c)\Gamma(e-d)}\left(\frac{(c)_2 (d)_2}{(e-c-d-2)_2} + \frac{3cd}{(e-c-d-1)} + 1\right).$$

iii. For $c, d > 0,\ e > c + d + 3$,

$$\sum_{l=0}^{\infty}(l+1)^3 \frac{(c)_l(d)_l}{(e)_l(1)_l} = \frac{\Gamma(e)\Gamma(e-c-d)}{\Gamma(e-c)\Gamma(e-d)}\left(\frac{(c)_3(d)_3}{(e-c-d-3)_3} + \frac{6(c)_2(d)_2}{(e-c-d-2)_2} + \frac{7cd}{(e-c-d-1)} + 1\right).$$

iv. For $c, d, e \neq 1$, with $e > \max\{0, c+d-1\}$,

$$\sum_{l=0}^{\infty} \frac{(c)_l(d)_l}{(e)_l(1)_{l+1}} = \frac{1}{(c-1)(d-1)}\left(\frac{\Gamma(e)\Gamma(e-c-d+1)}{\Gamma(e-c)\Gamma(e-d)} - (e-1)\right).$$

3. Main Results

In Theorem 3.1, we prove a one-to-one correspondence between the classes $\mathcal{R}_{\lambda}^{\delta}(L,M)$ and $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$.

Theorem 3.1: *Let the harmonic function* $\xi = s + \overline{u} \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$ *if and only if* $\mathcal{T}_{\mu} = s + \mu u \in \mathcal{R}_{\lambda}^{\delta}(L,M)$ *for each* $\mu(|\mu| = 1)$.

Proof: Suppose $\xi = s + \overline{u} \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$, for each $|\mu| = 1$, we have

$$\begin{aligned} &Re\{\mathcal{T}_{\mu}'(z) + \lambda z \mathcal{T}_{\mu}''(z) + \delta z^2 \mathcal{T}_{\mu}'''(z)\} \\ &= Re\{s'(z) + \lambda z s''(z) + \delta z^2 s'''(z) + \mu(u'(z) + \lambda z u''(z) + \delta z^2 u'''(z))\} \\ &> Re\{s'(z) + \lambda z s''(z) + \delta z^2 s'''(z)\} - \left|u'(z) + \lambda z u''(z) + \delta z^2 u'''(z)\right| \\ &> \left(\frac{L-1}{2M}\right), (z \in U). \end{aligned}$$

Thus, $\mathcal{T}_{\mu} \in \mathcal{R}_{\lambda}^{\delta}(L,M)$, for each $\mu(|\mu| = 1)$. Conversely, let $\mathcal{T}_{\mu} = s + \mu u \in \mathcal{R}_{\lambda}^{\delta}(L,M)$, then

$$\begin{aligned} &Re\{s'(z) + \lambda z s''(z) + \delta z^2 s'''(z)\} \\ &> Re(-\mu(u'(z) + \lambda z u''(z) + \delta z^2 u'''(z))) + \left(\frac{L-1}{2M}\right), (z \in U). \end{aligned}$$

Choosing $\mu(|\mu| = 1)$, we obtain

$$Re\left\{s'(z) + \lambda z s''(z) + \delta z^2 s'''(z) - \left(\frac{L-1}{2M}\right)\right\} > \left|u'(z) + \lambda z u''(z) + \delta z^2 u'''(z)\right|, (z \in U).$$

Hence $\xi \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$.

Theorem 3.2: *The function* $\mathcal{T} \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$ *is close-to-convex in U.*

Proof: In the Lemma 2.3, we obtained that $\mathcal{T}_{\mu} = s + \mu u \in \mathcal{R}_{\lambda}^{\delta}(L,M)$ is close-to-convex in U, $(|\mu| = 1)$.

Now in light of Lemma 2.1 and Theorem 3.1, we can prove function $\mathcal{T} \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$ close-to-convex in *U*.

In Theorem 3.3, we investigate coefficient bounds for functions in $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$

Theorem 3.3: *Let* $\xi = s + \overline{u} \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$, *then for* $l \geq 2$,

$$|b_l| \leq \frac{1 - \left(\frac{L-1}{2M}\right)}{l[1 + (l-1)(\lambda + \delta(l-2))]}.$$

The equality is hold for

$$\xi(z) = z + \frac{1-\left(\frac{L-1}{2M}\right)}{l[1+(l-1)(\lambda+\delta(l-2))]}\bar{z}^l.$$

Proof: Let $\xi = s + \bar{u} \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$. Now we use the series of $u(z)$, we obtain

$$\begin{aligned}
& r^{l-1}[l + l\lambda(l-1) + \delta l(l-1)(l-2)]|b_l| \\
& \leq \frac{1}{2\pi}\int_0^{2\pi} \left|u'(re^{i\theta}) + \lambda re^{i\theta}u''(re^{i\theta}) + \delta r^2 e^{2i\theta}u'''(re^{i\theta})\right| d\theta \\
& < \frac{1}{2\pi}\int_0^{2\pi} Re\left\{s'((re^{i\theta}) + \lambda re^{i\theta}s''(re^{i\theta}) + \delta r^2 e^{2i\theta}s'''(re^{i\theta}) - \left(\frac{L-1}{2M}\right)\right\} d\theta \\
& = \frac{1}{2\pi}\int_0^{2\pi} Re\left\{1 + \sum_{l=2}^{\infty}\Lambda(l,\lambda,\delta,\theta) - \left(\frac{L-1}{2M}\right)\right\} d\theta \\
& = 1 - \left(\frac{L-1}{2M}\right).
\end{aligned}$$

Where

$$\Lambda(l,\lambda,\delta,\theta) = [l + l\lambda(l-1) + \delta l(l-1)(l-2)]a_l r^{l-1}e^{i(l-1)\theta}.$$

Taking $r \to 1^-$, we get the desired bound.

Theorem 3.4: *Let* $\xi = s + \bar{u} \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$, *then for* $l \geq 2$, *we have*

i. $|a_l| + |b_l| \leq \dfrac{2\left(1-\left(\frac{L-1}{2M}\right)\right)}{l[1+(l-1)(\lambda+\delta(l-2))]}$,

ii. $\left||a_l| - |b_l|\right| \leq \dfrac{2\left(1-\left(\frac{L-1}{2M}\right)\right)}{l[1+(l-1)(\lambda+\delta(l-2))]}$,

iii. $|a_l| \leq \dfrac{2\left(1-\left(\frac{L-1}{2M}\right)\right)}{l[1+(l-1)(\lambda+\delta(l-2))]}$.

This result is sharp for the function

$$\xi(z) = z + \sum_{l=2}^{\infty}\frac{2\left(1-\left(\frac{L-1}{2M}\right)\right)}{l[1+(l-1)(\lambda+\delta(l-2))]}z^l.$$

Proof: Suppose that $\xi = s + \bar{u} \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$, then from Theorem 3.1, $\mathcal{T}_\mu = s + \mu u \in \mathcal{R}_{\lambda}^{\delta}(L,M)$ for each $\mu(|\mu|=1)$. Thus, for each $|\mu|=1$, we have

$$Re\{(s+\mu u)' + \lambda z(s+\mu u)'' + \delta z^2(s+\mu u)'''\} > \left(\frac{L-1}{2M}\right)$$

for $z \in U$ and there exists an analytic function

$$p(z) = 1 + \sum_{l=2}^{\infty} p_l z^l, \text{ and } Re(p(z) > 0 \text{ in } U,$$

such that

$$s'(z) + \lambda z s''(z) + \delta z^2 s'''(z) + \mu(u'(z) + \lambda z u''(z) + \delta z^2 u'''(z)) = \left(1 - \left(\frac{L-1}{2M}\right)\right) p(z). \tag{3.1}$$

Evaluating coefficients on both sides of (3.1), we have

$$l[1 + (l-1)(\lambda + \delta(l-2))](a_l + \mu b_l) = \left(1 - \left(\frac{L-1}{2M}\right)\right) p_{l-1}, l \geq 2.$$

Since for $|\mu| = 1$ and $l \geq 1$, we apply $|p_l| \leq 2$, we get the proof of (i). Similarly we can prove (ii) and (iii).

In the following result we find sufficient condition for a function belonging to class $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L, M)$.

Theorem 3.5: *Let* $\xi = s + \bar{u} \in \mathcal{H}^0$ *with*

$$\sum_{l=2}^{\infty} l[1 + (l-1)(\lambda + \delta(l-2))]\left(|a_l| + |b_l|\right) \leq 1 - \left(\frac{L-1}{2M}\right), \tag{3.2}$$

then, $\xi \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L, M)$.

Proof: Let $\xi = s + \bar{u} \in \mathcal{H}^0$. Then using (3.2),

$$\begin{aligned}
Re\left\{s'(z) + \lambda z s''(z) + \delta z^2 s'''(z) - \left(\frac{L-1}{2M}\right)\right\} &= Re\left\{1 - \left(\frac{L-1}{2M}\right) + \sum_{l=2}^{\infty} l[1 + (l-1)(\lambda + \delta(l-2))] a_l z^{l-1}\right\} \\
&> 1 - \left(\frac{L-1}{2M}\right) - \sum_{l=2}^{\infty} l[1 + (l-1)(\lambda + \delta(l-2))]\left|a_l\right| \\
&\geq \sum_{l=2}^{\infty} l[1 + (l-1)(\lambda + \delta(l-2))]\left|b_l\right| \\
&> \left|\sum_{l=2}^{\infty} l[1 + (l-1)(\lambda + \delta(l-2))] b_l z^{l-1}\right| \\
&= \left|u'(z) + \lambda z u''(z) + \delta z^2 u'''(z)\right|.
\end{aligned}$$

Hence $\xi \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L, M)$.

Corollary 3.6: *Let* $\xi = s + \bar{u} \in \mathcal{H}^0$. *If*

$$\sum_{l=2}^{\infty} l^2[3 - l + \lambda)(l-1)]\left(|a_l| + |b_l|\right) \leq 2,$$

then, $\xi \in \mathcal{R}_{\mathcal{H}}^{0}\left(\lambda, \frac{\lambda-1}{2}, 1, -1\right)$ with $\lambda \geq 1$.

Example 3.7: *By taking* $\lambda = 0.5$ *and* $\delta = 0.05$, $L = 1$, *and* $M = -1$ *in Theorem 3.5, the harmonic polynomials* $\xi_1(z) = z + 0.15\bar{z}^3$ *and* $\xi_1(z) = z - 0.079z^3 + 0.079\bar{z}^3$ *belong to* $\mathcal{R}_{\mathcal{H}}^{0,0.5,0.05}(1, -1)$.

Example 3.8: *Let for* $\lambda = 3$ *and* $\delta = 1$, $L = 1$, *and* $M = -1$ *in Theorem 3.5, the harmonic polynomials* $\xi_3(z) = z - \frac{1}{16}\overline{z}^{-2} + \frac{1}{54}\overline{z}^{-3}$ *and* $\xi_1(z) = z - 0.079z^3 + 0.079\overline{z}^{-3}$ *belong to* $\mathcal{R}_{\mathcal{H}}^{0,3,1}(1,-1)$.

Theorem 3.9: *Let* $\xi = s + \overline{u} \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$, $\lambda \geq \delta \geq 0$, $1 \leq L < M \leq -1$ *and* $M \neq 0$. *Then*

$$|z| + 2\left(1 - \left(\frac{L-1}{2M}\right)\right)\sum_{l=2}^{\infty}\frac{(-1)^{l-1}|z|^l}{\delta l^3 + (\lambda - 3\delta)l^2 + (1-\lambda+2\delta)l} \leq |\xi(z)|,$$

$$|\xi(z)| \leq |z| + 2\left(1 - \left(\frac{L-1}{2M}\right)\right)\sum_{l=2}^{\infty}\frac{|z|^l}{\delta l^3 + (\lambda - 3\delta)l^2 + (1-\lambda+2\delta)l}.$$

Result is sharp for the function

$$\xi(z) = z + \frac{1 - \left(\frac{L-1}{2M}\right)}{l[1 + (l-1)(\lambda + \delta(l-2))]}\overline{z}^{-l}.$$

Proof: Suppose that $\xi = s + \overline{u} \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$. Then by using Theorem 3.1, $\mathcal{T}_\mu \in \mathcal{R}_\lambda^\delta(L,M)$ for each $\mu(|\mu| = 1)$, we have

$$Re\Psi(z) > \left(\frac{L-1}{2M}\right),$$

where,

$$\Psi(z) = \mathcal{T}_\mu'(z) + \lambda z\mathcal{T}_\mu''(z) + \delta z^2\mathcal{T}_\mu'''(z).$$

Now by using the process of calculation of Rosihan et al. [22], we have

$$\begin{aligned}\Psi(z) &= \mathcal{T}_\mu'(z) + (\rho(1+\eta)+\eta)z\mathcal{T}_\mu''(z) + \rho\eta z^2\mathcal{T}_\mu'''(z)\\ &= \eta z^{1-\frac{1}{\eta}}\left(\rho z^{1+\frac{1}{\eta}}\mathcal{T}_\mu''(z) + z^{\frac{1}{\eta}}\mathcal{T}_\mu'(z)\right)'\\ &= \eta z^{1-\frac{1}{\eta}}\left(\rho z^{1+\frac{1}{\eta}-\frac{1}{\rho}}\left(z^{\frac{1}{\rho}}\mathcal{T}_\mu'(z)\right)'\right)',\end{aligned} \tag{3.3}$$

where,

$$\rho = \frac{(\lambda-\delta) - \sqrt{(\lambda-\delta)^2 - 4\delta}}{2},$$
$$\eta + \rho = \lambda - \delta, \rho\eta = \delta,$$
$$Re\rho \geq 0, Re\eta \geq 0.$$

Then integrating (3.3) gives

$$\rho z^{1+\frac{1}{\eta}-\frac{1}{\rho}}\left(z^{\frac{1}{\rho}}\mathcal{T}_\mu'(z)\right)' = \frac{1}{\eta}\int_0^z \upsilon^{\frac{1}{\eta}-1}\Psi(\upsilon)d\upsilon.$$

Making substitution $v = \theta^{\eta} z$ and after some simplification, we get

$$\left(z^{\frac{1}{\rho}} \mathcal{T}_{\mu}'(z) \right)' = \frac{1}{\rho} z^{\frac{1}{\rho}-1} \int_0^1 \Psi(\theta^{\eta} z) d\theta. \tag{3.4}$$

Now, integrating (3.4) and making substitution $\vartheta = v^{\rho} z$ gets

$$\begin{aligned} z^{\frac{1}{\rho}} \mathcal{T}_{\mu}'(z) &= \frac{1}{\rho} \int_0^z \vartheta^{\frac{1}{\rho}-1} \int_0^1 \Psi(\theta^{\eta} \vartheta) d\theta dv \\ &= z^{\frac{1}{\rho}} \int_0^1 \int_0^1 \Psi(\theta^{\eta} v^{\rho} z) d\theta dv, \end{aligned}$$

which simplifies to

$$\mathcal{T}_{\mu}'(z) = \int_0^1 \int_0^1 \Psi(\theta^{\eta} v^{\rho} z) d\theta dv. \tag{3.5}$$

Since

$$Re(\Psi(z)) > \frac{L-1}{2M},$$

then

$$\Psi(z) \prec \frac{1 + \left(1 - 2\left(\frac{L-1}{2M}\right)\right) z}{1-z}.$$

Let

$$\Psi(z) = 1 + \sum_{l=1}^{\infty} \frac{z^l}{(1+\eta l)(1+\rho l)} = \int_0^1 \int_0^1 \frac{d\theta dv}{1 - \theta^{\eta} v^{\rho} z},$$

and

$$Q(z) = \frac{1 + \left(1 - 2\left(\frac{L-1}{2M}\right)\right) z}{1-z} = 1 + \sum_{l=1}^{\infty} 2\left(1 - \frac{L-1}{2M}\right) z^l.$$

Then from (3.5), we have

$$\begin{aligned} \mathcal{T}_{\mu}'(z) \prec (\Psi * Q)(z) &= \left(1 + \sum_{l=1}^{\infty} \frac{z^l}{(1+\eta l)(1+\rho l)}\right) * \left(1 + \sum_{l=1}^{\infty} 2\left(1 - \frac{L-1}{2M}\right) z^l\right) \\ &= 1 + \sum_{l=1}^{\infty} \frac{2\left(1 - \frac{L-1}{2M}\right)}{1 + (\lambda - \delta) l + \delta l^2} z^l. \end{aligned}$$

Since

$$\begin{aligned} \left|\mathcal{T}_{\mu}'(z)\right| &= \left|s'(z) + \mu u'(z)\right| \\ &\le 1 + 2\left(1 - \frac{L-1}{2M}\right) \sum_{l=1}^{\infty} \frac{|z|^l}{1 + (\lambda - \delta) l + \delta l^2} \end{aligned}$$

and

$$\begin{aligned}\left|\mathcal{T}_{\mu}'(z)\right| &= \left|s'(z)+\mu u'(z)\right| \\ &\geq 1+2\left(1-\frac{L-1}{2M}\right)\sum_{l=1}^{\infty}\frac{(-1)^{l}\left|z\right|^{l}}{1+(\lambda-\delta)l+\delta l^{2}}.\end{aligned}$$

In particular, we have

$$\left|s'(z)\right|+\left|u'(z)\right| \leq 1+2\left(1-\frac{L-1}{2M}\right)\sum_{l=1}^{\infty}\frac{|z|^{l}}{1+(\lambda-\delta)l+\delta l^{2}},$$

$$\left|s'(z)\right|-\left|u'(z)\right| \geq 1+2\left(1-\frac{L-1}{2M}\right)\sum_{l=1}^{\infty}\frac{(-1)^{l}\,|z|^{l}}{1+(\lambda-\delta)l+\delta l^{2}}.$$

Let the radial segment Γ from 0 to z, then

$$\begin{aligned}\left|\xi(z)\right| &= \left|\int_{\Gamma}\left(\frac{\partial\xi}{\partial\varepsilon}d\varepsilon+\frac{\partial\xi}{\partial\overline{\varepsilon}}d\overline{\varepsilon}\right)\right| \\ &\leq \int_{\Gamma}\left(\left|s'(\varepsilon)\right|+\left|u'(\varepsilon)\right|\right)\left|d\varepsilon\right| \\ &\leq \int_{0}^{|z|}\left(1+2\left(1-\frac{L-1}{2M}\right)\sum_{l=1}^{\infty}\frac{\left|t\right|^{l}}{1+(\lambda-\delta)l+\delta l^{2}}\right)dt \\ &= \left|z\right|+2\left(1-\frac{L-1}{2M}\right)\sum_{l=1}^{\infty}\frac{\left|z\right|^{l+1}}{(l+1)[1+(\lambda-\delta)l+\delta l^{2}]} \\ &= \left|z\right|+2\left(1-\frac{L-1}{2M}\right)\sum_{l=2}^{\infty}\frac{\left|z\right|^{l}}{\delta l^{3}+(\lambda-3\delta)l^{2}+(1-\lambda+2\delta)l},\end{aligned}$$

and

$$\begin{aligned}\left|\xi(z)\right| &\geq \int_{\Gamma}\left(\left|s'(\varepsilon)\right|-\left|u'(\varepsilon)\right|\right)\left|d\varepsilon\right| \\ &\leq \int_{0}^{|z|}\left(1+2\left(1-\frac{L-1}{2M}\right)\sum_{l=1}^{\infty}\frac{(-1)^{l}\left|t\right|^{l}}{1+(\lambda-\delta)l+\delta l^{2}}dt\right) \\ &= \left|z\right|+2\left(1-\frac{L-1}{2M}\right)\sum_{l=2}^{\infty}\frac{(-1)^{l-1}\left|z\right|^{l}}{\delta l^{3}+(\lambda-3\delta)l^{2}+(1-\lambda+2\delta)l}.\end{aligned}$$

Hence complete the proof.

Remark 3.10: *The results which have been derived in this section, yield the results of the classes* $\mathcal{R}_{\mathcal{H}}^{0,0,0}(1,-1)$ *and* $\mathcal{R}_{\mathcal{H}}^{0,\lambda,0}(1,-1)$ *which are defined and studied in the following papers* [8, 4, 5, 3].

Theorem 3.11: *The class* $\mathcal{R}_{\mathcal{H}}^{0,\lambda,0}(L,M)$ *is closed under convex combinations.*

Proof: Let we have

$$\xi_{i}=s_{i}+\overline{u_{i}}\in\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$$

for $i = 1, 2, \ldots, l,$ and

$$\sum_{i=1}^{l} \tau_i = 1, (0 \le \tau_i \le 1).$$

The convex combination for ξ_i can be written as:

$$\xi(z) = \sum_{i=1}^{l} \tau_i \xi_i(z) = s(z) + \overline{u(z)},$$

where

$$s(z) = \sum_{i=1}^{l} \tau_i s_i(z) \text{ and } u(z) = \sum_{i=1}^{l} \tau_i u_i(z).$$

where s and u are analytic in U with

$$s(0) = u(0) = s'(0) - 1 = u'(0) = 0$$

and

$$\begin{aligned} Re\{s'(z) + \lambda z s''(z) + \delta z^2 s'''(z)\} &= Re\left\{\sum_{i=1}^{l} \tau_i (s_i'(z) + \lambda z s_i''(z) + \delta z^2 s_i'''(z))\right\} \\ &> \sum_{i=1}^{l} \tau_i \left| u_i'(z) + \lambda z u_i''(z) + \delta z^2 u_i'''(z) \right| \\ &\ge \left| u'(z) + \lambda z u''(z) + \delta z^2 u'''(z) \right|. \end{aligned}$$

This shows that $\xi \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L, M).$

In Theorem 3.12, we use Lemma 2.7 and prove that the class $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L, M)$ is closed under convolutions.

Theorem 3.12: *Let $\xi \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L, M)$, for $i = 1, 2.$ Then*

$$\xi_1 * \xi_2 \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L, M).$$

Proof: Suppose $\xi_i = s_i + \overline{u_i} \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L, M)$ $(i = 1, 2).$ Then $\xi_1(z)$ and $\xi_2(z)$ can be written as

$$\xi_1 * \xi_2 = s_1 * s_2 + \overline{u_1 * u_2}.$$

In order to prove that

$$\xi_1 * \xi_2 \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L, M),$$

we have to prove that

$$\mathcal{T}_\mu = s_1 * s_2 + \mu(u_1 * u_2) \in \mathcal{R}_\lambda^\delta(L, M),$$

for each $\mu(|\mu| = 1).$ By Lemma 2.7, the class $\mathcal{R}_\lambda^\delta(L, M)$ is closed under convolutions for each $\mu(|\mu| = 1),$

$$s_i + \mu u_i \in \mathcal{R}_\lambda^\delta(L, M).$$

Then both $\mathcal{T}_1$ and $\mathcal{T}_1$ given by

$$\mathcal{T}_1 = (s_1 - u_1) * (s_2 - \mu u_2)$$

and

$$\mathcal{T}_2 = (s_1 + u_1) * (s_2 + \mu u_2),$$

and $\mathcal{T}_1, \mathcal{T}_1 \in \mathcal{R}_\lambda^\delta(L,M)$. We know that $\mathcal{R}_\lambda^\delta(L,M)$ is closed under convex combinations.
Then the function

$$\mathcal{T} = \frac{1}{2}(\mathcal{T}_1 + \mathcal{T}_2) = s_1 * s_2 + \mu(u_1 * u_2)$$

belongs to $\mathcal{R}_\lambda^\delta(L,M)$. Hence $\mathcal{R}_\mathcal{H}^{0,\lambda,\delta}(L,M)$ is closed under convolution.

Theorem 3.13: *Let $\xi \in \mathcal{R}_\mathcal{H}^{0,\lambda,\delta}(L,M)$ and $\phi \in \mathcal{A}$ be such that*

$$Re\left\{\frac{\phi(z)}{z}\right\} > \frac{1}{2},$$

for $z \in U$. Then

$$\xi * \phi \in \mathcal{R}_\mathcal{H}^{0,\lambda,\delta}(L,M).$$

Proof. Suppose $\xi = s + \overline{u} \in \mathcal{R}_\mathcal{H}^{0,\lambda,\delta}(L,M)$. Then

$$\mathcal{T}_\mu = s + \mu u \in \mathcal{R}_\lambda^\delta(L,M)$$

for each $\mu(|\mu| = 1)$. In order to show that

$$\xi * \phi \in \mathcal{R}_\mathcal{H}^{0,\lambda,\delta}(L,M),$$

we need to show that

$$\mathcal{G} = s * \phi + \mu(u * \phi) \in \mathcal{R}_\lambda^\delta(L,M)$$

for each $\mu(|\mu| = 1)$. Write $\mathcal{G} = \mathcal{T}_\mu * \phi$, and

$$\frac{1}{1-\left(\frac{L-1}{2M}\right)}\left(\mathcal{G}'(z) + \lambda z\mathcal{G}''(z) + \delta z^2\mathcal{G}'''(z) - \left(\frac{L-1}{2M}\right)\right)$$
$$= \frac{1}{1-\left(\frac{L-1}{2M}\right)}\left(\mathcal{T}_\mu'(z) + \lambda z\mathcal{T}_\mu''(z) + \delta z^2\mathcal{T}_\mu'''(z) - \left(\frac{L-1}{2M}\right)\right) * \frac{\phi(z)}{z}.$$

Since $Re\left(\frac{\phi(z)}{z}\right) > \frac{1}{2}$ and

$$Re\left(\mathcal{T}_\mu'(z) + \lambda z\mathcal{T}_\mu''(z) + \delta z^2\mathcal{T}_\mu'''(z) - \left(\frac{L-1}{2M}\right)\right) > 0$$

in U. Using the Lemma 2.5, Hence $\mathcal{G} \in \mathcal{R}_\lambda^\delta(L,M)$.

Corollary 3.14: *Let $\xi \in \mathcal{R}_\mathcal{H}^{0,\lambda,\delta}(L,M)$ and $\phi \in \mathcal{K}$, then $\xi * \phi \in \mathcal{R}_\mathcal{H}^{0,\lambda,\delta}(L,M)$.*

Proof. Suppose $\phi \in \mathcal{K}$, then $Re\left(\frac{\phi(z)}{z}\right) > \frac{1}{2}$ for $z \in U$. Theorem 3.13 concludes that $\xi * \phi \in \mathcal{R}_\mathcal{H}^{0,\lambda,\delta}(L,M)$.

4. Applications

In this section, we involve Gaussian hypergeometric function to discuss some applications of newly defined class of harmonic functions and construct harmonic polynomials which belong to the considered class $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$.

Theorem 4.1: *Suppose*

$$\xi_1(z) = z + \overline{zF(c,d,e;z)},$$
$$\xi_2(z) = z + \overline{z(F(c,d,e;z)-1)},$$

and

$$\xi_3(z) = z + \overline{z\int_0^z F(c,d,e;t)dt}$$

where c is a positive real number and either $c,d \in (-1,\infty)$ with $cd > 0$ or $c,d \in \mathbb{C}\backslash\{0\}$ with $d = \bar{c}$.

i. If $Re(c+d)+3<e$ and

$$\left[2(1+\lambda)+(1+4\lambda+6\delta)\frac{cd}{(e-c-d-1)}+(\lambda+6\delta)\frac{(c)_2(d)_2}{(e-c-d-2)_2}+\delta\frac{(c)_3(d)_3}{(e-c-d-3)_3}\right] F(c,d,e;1) \le 1-\left(\frac{L-1}{2M}\right) \tag{4.1}$$

then, $\xi_1 \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$.

ii. If $Re(c+d)+3<e$ and

$$\left[1+(1+2\lambda)\frac{cd}{(e-c-d-1)}+(\lambda+3\delta)\frac{(c)_2(d)_2}{(e-c-d-2)_2}+\delta\frac{(c)_3(d)_3}{(e-c-d-3)_3}\right] F(c,d,e;1) \le 2-\left(\frac{L-1}{2M}\right) \tag{4.2}$$

then $\xi_2 \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$.

iii. If $c,d,e \neq 1$ and $Re(c+d)+2<e$

$$\left(\begin{array}{l}\frac{e-c-d}{(c-1)(d-1)}+1+2\lambda+(\lambda+3\delta)\\ \frac{cd}{(e-c-d-1)}+\delta\frac{(c)_2(d)_2}{(e-c-d-2)_2}\end{array}\right)F(c,d,e;1) \le 1+\frac{e-1}{(c-1)(d-1)}-\left(\frac{L-1}{2M}\right), \tag{4.3}$$

then $\xi_3 \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$.

Proof: (i) Suppose $\xi_1(z) = z + \overline{zF(c,d,e;z)} = z + \overline{\sum_{l=2}^{\infty} Q_{1,l} z^l}$, where

$$Q_{1,l} = \frac{(c)_{l-2}(d)_{l-2}}{(e)_{l-2}(l-1)!}, l \ge 2.$$

In view of Theorem 3.5, to prove that $\xi_1 \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$, we need to show that

$$\sum_{l=2}^{\infty} l[1+(l-1)(\lambda+\delta(l-2))]\left|Q_{1,l}\right| \leq \left(1-\left(\frac{L-1}{2M}\right)\right).$$

Now by the using Lemma 2.8 and Gauss formula (1.5) we have

$$\begin{aligned}
&\sum_{l=2}^{\infty} l[1+(l-1)(\lambda+\delta(l-2))]\left|Q_{1,l}\right| \\
&=\sum_{l=2}^{\infty} l\left[1+(l-1)(\lambda+\delta(l-2))\right]\frac{(c)_{l-2}(d)_{l-2}}{(e)_{l-2}(l-2)!}, \\
&=\sum_{l=0}^{\infty}(l+2)[1+(l+1)(\lambda+\delta l)]\frac{(c)_l(d)_l}{(e)_l(l)!}, \\
&=\sum_{l=0}^{\infty}\frac{(c)_l(d)_l}{(e)_l(l)!}+(1+\lambda-\delta)\sum_{l=0}^{\infty}(l+1)\frac{(c)_l(d)_l}{(e)_l(l)!} \\
&\quad+\lambda\sum_{l=0}^{\infty}(l+1)^2\frac{(c)_l(d)_l}{(e)_l(l)!}+\delta\sum_{l=0}^{\infty}(l+1)^3\frac{(c)_l(d)_l}{(e)_l(l)!}, \\
&=F(c,d,e;1)+(1+\lambda-\delta)\left(\frac{cd}{e-c-d-1}+1\right)F(c,d,e;1) \\
&\quad+\lambda\left(\frac{(c)_2(d)_2}{(e-c-d-2)_2}+\frac{3cd}{(e-c-d-1)}+1\right)F(c,d,e;1) \\
&\quad+\delta\left(\frac{(c)_3(d)_3}{(e-c-d-3)_3}+\frac{6(c)_2(d)_2}{(e-c-d-2)_2}+\frac{7cd}{e-c-d-1}+1\right)F(c,d,e;1).
\end{aligned}$$

Hence, condition (4.1) concludes that $\xi_1 \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$.

(ii) Suppose $\xi_2(z)=z+\overline{z(F(c,d,e;z)-1)}=z+\overline{\sum_{l=2}^{\infty}Q_{2,l}z^l}$, where

$$Q_{2,l}=\frac{(c)_{l-1}(d)_{l-1}}{(e)_{l-1}(l-1)!}, l\geq 2.$$

In view of Theorem 3.5, to prove that $\xi_2 \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$, we need to show that

$$\sum_{l=2}^{\infty} l[1+(l-1)(\lambda+\delta(l-2))]\left|Q_{2,l}\right| \leq \left(1-\left(\frac{L-1}{2M}\right)\right).$$

Now by the using Lemma 2.8 and Gauss formula (1.5) we have

$$\begin{aligned}
&\sum_{l=2}^{\infty} l[1+(l-1)(\lambda+\delta(l-2))]\left|Q_{2,l}\right| \\
&=\sum_{l=0}^{\infty}\frac{(c)_{l+1}(d)_{l+1}}{(e)_{l+1}(l+1)!}+(1+2\lambda)\sum_{l=0}^{\infty}\frac{(c)_{l+1}(d)_{l+1}}{(e)_{l+1}(l)!} \\
&\quad+(\lambda+3\delta)\sum_{l=1}^{\infty}\frac{(c)_{l+1}(d)_{l+1}}{(e)_{l+1}(l-1)!}+\delta\sum_{l=2}^{\infty}\frac{(c)_{l+1}(d)_{l+1}}{(e)_{l+1}(l-2)!},
\end{aligned}$$

$$= F(c,d,e;1) - 1 + (1+2\lambda)\frac{cd}{e} + F(c+1,d+1,e+1;1)$$
$$+(\lambda+3\delta)\frac{(c)_2(d)_2}{(e)_2}F(c+2,d+2,e+2;1)$$
$$+\delta\frac{(c)_3(d)_3}{(e)_3}F(c+3,d+3,e+3;1).$$

Condition (4.2) concludes that $\xi_2 \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$.

(iii) Suppose $\xi_3(z) = z + \overline{z\int_0^z F(c,d,e;t)dt} = z + \overline{\sum_{l=2}^{\infty} Q_{3,l} z^l}$, where

$$Q_{3,l} = \frac{(c)_{l-2}(d)_{l-2}}{(e)_{l-2}(l-1)!}, l \geq 2.$$

In view of Theorem 3.5, to prove that $\xi_3 \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$, we need to show

$$\sum_{l=2}^{\infty} l[1+(l-1)(\lambda+\delta(l-2))]\left|Q_{3,l}\right| \leq \left(1-\left(\frac{L-1}{2M}\right)\right).$$

Now by the using Lemma 2.8 and Gauss formula (1.5) we have

$$\sum_{l=2}^{\infty} l[1+(l-1)(\lambda+\delta(l-2))]\left|Q_{3,l}\right|$$
$$= \sum_{l=2}^{\infty} l[1+(l-1)(\lambda+\delta(l-2))]\frac{(c)_{l-2}(d)_{l-2}}{(e)_{l-2}(l-1)!},$$
$$= \sum_{l=0}^{\infty}\frac{(c)_l(d)_l}{(e)_l(l+1)!} + (1+2\lambda)\sum_{l=0}^{\infty}\frac{(c)_l(d)_l}{(e)_l(l)!}$$
$$+\left(\lambda+3\delta\right)\sum_{l=1}^{\infty}\frac{(c)_l(d)_l}{(e)_l(l-1)!} + \delta\sum_{l=2}^{\infty}\frac{(c)_l(d)_l}{(e)_l(l-2)!},$$
$$= \frac{e-1}{(c-1)(d-1)}[F(c-1,d-1,e-1;1)-1]$$
$$+(1+2\lambda)F(c,d,e;1) + (\lambda+3\delta)\frac{cd}{(e-c-d-1)}F(c,d,e;1)$$
$$+\delta\frac{(c)_2(d)_2}{(e-c-d-2)_2}F(c,d,e;1).$$

Condition (4.3) concludes that $\xi_3 \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$.

Conclusion

In this paper, we defined a new class $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$ of normalized harmonic functions in the open unit disk U which is satisfying third-order differential inequality and investigated some new and know lemmas to prove our main results for this class. Then Theorem 3.1 proved one-to-one correspondence between the classes $\mathcal{R}_{\lambda}^{\delta}(L,M)$ and $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$ and Theorem 3.2 proved that every function $f \in \mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$ is closed-to-convex in open unit disk U. Furthermore, we examined various properties of the this class $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$, such as coefficient bounds, growth estimates, sufficient coefficient condition. We established that class $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$ is closed under convex combination and convolution. In Theorem 4.1, we involved Gaussian hypergeometric function and construct harmonic polynomials which belong to the considered class $\mathcal{R}_{\mathcal{H}}^{0,\lambda,\delta}(L,M)$.

References

[1] Lewy, H., *On the non-vanishing of the Jacobian in certain one-to-one mappings*. Bulletin of the American Mathematical Society, 42, (1983), 689–692. https://doi.org/10.1090/S0002-9904-1936-06397-4

[2] Clunie, J., Sheil-Small, T., *Harmonic univalent functions*. Annales Academiae Scientiarum Fennicae Series A I Mathematica 9, (1984), 3–25.

[3] Ponnusamy, S., Yamamoto, H., Yanagihara, H., *Variability regions for certain families of harmonic univalent mappings*. Complex Variables and Elliptic Equations, 58 (1), (2013), 23–34. https://doi.org/10.1080/17476933.2010.551200

[4] Li, L., Ponnusamy, S., *Injectivity of sections of univalent harmonic mappings*. Nonlinear Analysis, 89, (2013), 276–283. https://doi.org/10.1016/j.na.2013.05.016

[5] Li, L., Ponnusamy, S., *Disk of convexity of sections of univalent harmonic functions*. Journal of Mathematical Analysis and Applications, 408, (2013), 589–596. https://doi.org/10.1016/j.jmaa.2013.06.021

[6] Nagpal, S., Ravichandran, V., *Construction of subclasses of univalent harmonic mappings*. Journal of the Korean Mathematical Society, 53, (2014), 567–592. https://doi.org/10.4134/JKMS.2014.51.3.567

[7] Chichra, P. N., *New subclasses of the class of close-to-convex functions*. Proceedings of the American Mathematical Society, 62 (1), (1976), 37–43. https://doi.org/10.1090/S0002-9939-1977-0425097-1

[8] Ghosh, N., Vasudevarao, A., *On a subclass of harmonic close-to-convex mappings*. Monatshefte für Mathematik, 188, (2019), 247–267. https://doi.org/10.1007/s00605-017-1138-7

[9] Rajbala, C., Prajapat, J. K., *On a subclass of close-to-convex harmonic mappings*. Asian-European Journal of Mathematics, 14 (6), (2021), 2150102. https://doi.org/10.1142/S1793557121501023

[10] Yaşar, E, Yalçın, S. *Close-to-convexity of a class of harmonic mappings defined by a third-order differential inequality*. Turkish journal of. Mathematics, 45 (2), (2021), 678–694. https://doi.org/10.3906/mat-2004-50

[11] Dorff, M., *Convolutions of planar harmonic convex mappings*, Complex Variable, Theory Applications, 5 (3), (2001), 263–271. https://doi.org/10.1080/17476930108815381

[12] Nagpal, S., Ravichandran, V., *A subclass of close-to-convex harmonic mappings*. Complex variables and Elliptic Equations, 59 (2), (2014), 204–216. https://doi.org/10.1080/17476933.2012.727409

[13] Ponnusamy, S., Kaliraj, A. S., Starkov, V. V., *Absolutely convex, uniformly starlike and uniformly convex harmonic mappings*. Complex Variables and Elliptic Equations, 61 (10), (2016), 1418–1433. https://doi.org/10.1080/17476933.2016.1182518

[14] Goodloe, M., *Hadamard products of convex harmonic mappings*. Complex Variables Theory and Applications, 47 (2), (2002), 81–92. https://doi.org/10.1080/02781070290010841

[15] Duren, P., *Univalent functions*. in: Grundlehren Der Mathematischen Wissenschaften, vol. 259, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, (1983).

[16] Temme, N. M., *Special functions*. An Introduction to the Classical Functions of Mathematical Physics. New York, 694 NY, USA: Wiley, (1996).

[17] Miller, S. S., Mocanu P. T., *Differential subordinations and univalent functions*. Michigan Mathematical Journal, 28, (1981), 157–171. https://doi.org/10.1016/0022-0396(87)90146-X

[18] Miller, S. S., Mocanu, P. T., *Differential subordinations*. Theory and Applications. New York, NY, USA: Marcel Dekker, (1999).

[19] Fejér, L., *Über die positivitat von summen, die nach trigonometrischen order Legendreschen funktionen fortschreiten*. Acta Scientiarum Mathematicarum (1925), 75–86 (in German).

[20] Singh, R., Singh, S., *Convolution properties of a class of starlike functions*. Proceedings of American Mathematical Society, 106, (1989), 145–152. https://doi.org/10.2307/2047386

[21] Kim, Y. C., Ponnusamy, S., *Sufficiency for Gaussian hypergeometric functions to be uniformly convex*. International Journal of Mathematics and Mathematical Sciences, 22 (4), (1999), 765–773. https://doi.org/10.1155/S0161171299227652

[22] Rosihan, M. A., Lee, S. K., Subramanian, K. G., Swaminathan, A., *A third-order differential equation and starlikeness of a double integral operator*, Abstract and Applied Analysis, (2011). https://doi.org/10.1155/2011/901235.