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Topological-like notions via Δ-open sets

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Abstract

We provide a characterization of Δ-open and Δ-closed sets in topological spaces. Besides, based on the concepts of Δ-open and Δ-closed sets we investigate the notions of Δ-interior, Δ-exterior, Δ-closure, Δ-derived sets, Δ-boundary sets, and Δ-dense sets. Moreover, we study Δ-open and Δ-closed sets in product topology and subspace typology.

Keywords: open set closed set symmetric difference Δ-open set Δ-closed set

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1. Introduction

Several notions of open-like and closed-like sets in topological spaces were introduced and studied. The beginning was with Norman Levine who initiated the notion of *semi open sets*, [12]. After that α*-sets* and β*-sets* in topological spaces were introduced and considered as "nearly open" by Olav Njåstad, [17]. Furthermore, the notions of θ*-open* and δ*-open* sets were initiated, [24]. Later the concepts of *pre-open sets* and *semi-preopen sets* were started, [15] and [6], respectively.

The concept of closed sets in topological spaces was extended to *generalized closed sets*, [13]. However, another extension of closed sets in topological spaces called *semi generalized closed sets* was obtained in [7]. N. Palaniappan and K. Chandrasekhara Rao introduced *regular generalized closed sets*, [19]. Additionally, a class of closed-like sets called ψ*-closed* sets was established by M. Veera Kumar, [23].

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Recently, the idea of introducing new classes of open-like and closed-like sets in topological spaces is still attracting many researchers, for example [1–5], [8], [11], [14], [16], [20], [21], and [22].

A set in a topological space is called Δ-*open* if it is the symmetric difference of two open sets. The notion of Δ-open sets appeared in [18] and in [10]. However, it was pointed out in [18] and in [10] that the notion of Δ-open sets is due to a preprint by M. Veera Kumar. The complement of a Δ-open set is Δ-*closed*.

In Section 2, we establish a characterization for Δ-open (respectively, Δ-closed) sets that is free of the symmetric difference operation, Theorem 2.2 (respectively, Corollary 2.3). As a consequence, we see that a finite intersection (union) of Δ-open (Δ-closed) sets is Δ-open (Δ-closed).

In Section 3, notions of Δ-interior, Δ-exterior, and Δ-closure are investigated. Characterizations and properties of these notions are introduced as well.

Concepts and properties of Δ-limit points, Δ-boundary points, and Δ-dense sets are presented in Sections 4 and 5.

Section 6 is devoted to examine Δ-open and Δ-closed sets in the product topology. It is shown that a product of two Δ-open sets is again a Δ-open in the product topology, Theorem 6.2. However, this is not the case for Δ-closed sets as illustrated by Example 6.5.

Finally in Section 7 we consider Δ-open sets, Δ-closed sets, Δ-interior, and Δ-closure in subspace topology. In particular, characterizations of Δ-open and Δ-closed sets in subspace topology are given, Propositions 7.1 and 7.2.

2. Δ-open and Δ-closed sets

In this section we provide basic notions and results related to Δ-open and Δ-closed sets. These results will be used and applied in the subsequent sections.

Recall that for sets A and B their *symmetric difference* is given as $A\Delta B := (A-B)\cup(B-A) = (A\cup B)-(A\cap B)$.

Definition 2.1. ([18] and [10]) *A set A in a topological space* (X,τ) *is called* Δ-*open if there are open sets* O_1 *and* O_2 *so that* $A = O_1\Delta O_2$.

For an open set O in a topological space (X,τ), it is obvious that $O = O\Delta\phi$, so every open set is Δ-open. However, the set $(0,1]\cup[2,3) = (0,2)\Delta(1,3)$ is Δ-open set which is not open in the standard topology on $\mathbb{R}$. The complement of a Δ-open set is called Δ-*closed*. We provide a characterization of Δ-open sets.

Theorem 2.2. *A set A in a topological space* (X,τ) *is* Δ-*open if and only if there are an open set O and a closed set C such that* $A = O\cap C$.

Proof. Assume A is Δ-open, then there are two open sets O_1 and O_2 such that $A = O_1\Delta O_2 = (O_1\cup O_2)-(O_1\cap O_2) = (O_1\cup O_2)\cap(O_1\cap O_2)^c$. The conclusion follows directly by letting $O = O_1\cup O_2$ and $C = (O_1\cap O_2)^c$.

Conversely, suppose $A = O\cap C$ for some open set O and a closed set C. Then it is easily seen that $A = O\Delta(O\cap C^c)$. Hence, A is Δ-open.

Corollary 2.3. *A set B in a topological space* (X,τ) *is* Δ-*closed if and only if there are an open set O and a closed set C such that* $B = O\cup C$.

It follows from Theorem 2.2 that any open set and any closed set is Δ-open. Moreover, a finite intersection of Δ-open sets is Δ-open. However, union of two Δ-open sets need not be Δ-open and arbitrary intersection of Δ-open sets need not be Δ-open.

In the same manner, it follows from Corollary 2.3 that any open set and any closed set is Δ-closed. Additionally, a finite union of Δ-closed sets is Δ-closed. Nonetheless, intersection of two Δ-closed sets need not be Δ-closed and arbitrary union of Δ-closed sets need not be Δ-closed.

Example 2.4. *Let* $X = \{a,b,c,d,e\}$ *with a topology*

$$\tau = \{\phi, X, \{a,b,c\}, \{a,b,c,d\}\}.$$

The collection of all Δ-open sets in X is

$$\tau_{\Delta o} = \{\phi, X, \{a,b,c\}, \{a,b,c,d\}, \{d,e\}, \{d\}, \{e\}\}.$$

Clearly, the sets $\{a,b,c\}$ and $\{e\}$ are Δ-open, whereas their union is not.

It is worth noting from the previous example that the collection of all Δ-open sets does not form a topology in general.

Example 2.5. *Let* $\mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\}$ *be an enumeration of the rationals. For each* $n \in \mathbb{N}$*, let* $S_n = \mathbb{R} - \{r_1, r_2, \ldots, r_n\}$*. Then considering* $\mathbb{R}$ *under the standard topology, each* S_n *is open set and so it is Δ-open. Yet,* $\bigcap_{n=1}^{\infty} S_n = \mathbb{R} - \mathbb{Q}$ *is not Δ-open.*

It should be pointed out that the open set O and the closed set C in Corollary 2.3 can be chosen to be disjoint. Indeed, if $B = O \cup C$, with O is open and C is closed, then $B = O \cup (C - O)$ where $C - O$ is closed.

A topological space is said to be $T_{\frac{1}{2}}$ if each g-closed set is closed, [13]. The $T_{\frac{1}{2}}$-spaces were first defined by Norman Levine, [13]. Hereinafter, William Dunham obtained an interesting characterization of $T_{\frac{1}{2}}$-spaces independent of g-closedness. A topological space is $T_{\frac{1}{2}}$ if and only if each singleton is either open or closed, [9]. Obviously, a T_1-space is $T_{\frac{1}{2}}$.

Proposition 2.6. *If* (X,τ) *is a* $T_{\frac{1}{2}}$*-space, then singletons are Δ-open.*

Proof. Let $x \in X$. Then $\{x\}$ is either open or closed and $\{x\} = \{x\} \cap X$. Therefore, by Theorem 2.2 $\{x\}$ is Δ-open.

Corollary 2.7. *If* (X,τ) *is a* $T_{\frac{1}{2}}$*-space, then for* $x \in X$ *the set* $X - \{x\}$ *is Δ-closed.*

The converse of Proposition 2.6 need not be true.

Example 2.8 *Let* $X = \{a,b,c\}$ *with a topology* $\tau = \{\phi, X, \{a\}, \{a,b\}\}$*. The collection of all Δ-open sets is* $\tau_{\Delta o} = \{\phi, X, \{a\}, \{a,b\}, \{b,c\}, \{c\}, \{b\}\}$*. Then singletons are Δ-open, though X is not* $T_{\frac{1}{2}}$*.*

3. Δ-interior and Δ-closure

Let (X,τ) be a topological space and $A \subseteq X$. Recall that the union of all open sets contained in A is known as the *interior* of A and is is denoted by $Int(A)$. The *exterior* of a set is the interior of it complement, and it is denoted by $Ext(A)$. Besides, the intersection of all closed sets containing A is known as the *closure* of A and is is denoted by $Cl(A)$. Likewise we have the following notions.

Definition 3.1. *Let* (X,τ) *be a topological space and* $A \subseteq X$.

- (*i*). *The union of all Δ-open sets contained in A is said to be the Δ-interior of A and is denoted by* $\Delta Int(A)$.
- (*ii*). *The Δ-exterior of A is the Δ-interior of* $X - A$ *and is denoted by* $\Delta Ext(A)$*. So,* $\Delta Ext(A) = \Delta Int(X - A)$.
- (*iii*). *The intersection of all Δ-closed sets containing A is said to be the Δ-closure of A and is denoted by* $\Delta Cl(A)$.

Clearly, $\Delta Int(A)$ need not be Δ-open and $\Delta Cl(A)$ need not be Δ-closed. It should be also noted that if A is Δ-open, then $\Delta Int(A) = A$, and if A is Δ-closed, then $\Delta Cl(A) = A$. In either case the converse is not true.

Example 3.2. *Let* $\mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\}$ *be an enumeration of the rationals. For each* $n \in \mathbb{N}$, *let* $O_n = (-n,n)$ *and* $C_n = \{r_1, r_2, \ldots, r_n\}$. *Then* $A_n = O_n \cap C_n$ *is* Δ*-open set in* $\mathbb{R}$ *under the standard topology. Let* $A = \bigcup_{n=1}^{\infty} A_n$. *Then* $\Delta Int(A) = A$, *nevertheless* $A = \mathbb{Q}$ *is not* Δ*-open.*

Because each open set is Δ-open and each closed set is Δ-closed, the following result follows directly.

Lemma 3.3. *Let* (X,τ) *be a topological space and* $A \subseteq X$. *Then*

(*i*) $\Delta Int(A) \subseteq \Delta Cl(A)$.
(*ii*) $Int(A) \subseteq \Delta Int(A) \subseteq A$.
(*iii*) $A \subseteq \Delta Cl(A) \subseteq Cl(A)$.

We state and prove the following result on a relation between $\Delta Int(A)$ and $\Delta Cl(A)$.

Proposition 3.4. *Let* (X,τ) *be a topological space and* $A \subseteq X$. *Then*

(*i*) $X - \Delta Int(A) = \Delta Cl(X - A)$.
(*ii*) $X - \Delta Cl(A) = \Delta Int(X - A) = \Delta Ext(A)$.

Proof.

(*i*) Let $\mathcal{O} = \{O : O \text{ is } \Delta-open, \text{ and } O \subseteq A\}$. Then $X - \Delta Int(A) = X - \bigcup \mathcal{O} = \bigcap (X - \mathcal{O}) = \Delta Cl(X - A)$, where $X - \mathcal{O} = \{X - O : X - O \text{ is } \Delta-closed, \text{ and } X - A \subseteq X - O\}$.

(*ii*) Let $\mathcal{C} = \{C : C \text{ is } \Delta-closed, \text{ and } A \subseteq C\}$. Then $X - \Delta Cl(A) = X - \bigcup \Delta\mathcal{C} = \bigcap \Delta(X - \mathcal{C}) = \Delta Int(X - A)$ $= \Delta Ext(A)$, where $X - \mathcal{C} = \{X - C : X - C \text{ is } \Delta-open, \text{ and } X - C \subseteq X - A\}$.

A combination of Lemma 3.3 and Proposition 3.4 yields the next result.

Lemma 3.5. *Let* (X,τ) *be a topological space and* $A \subseteq X$. *Then* $Ext(A) \subseteq \Delta Ext(A) \subseteq X - A$.

Definition 3.6. *Let* (X,τ) *be a topological space and* $x \in X$. *A* Δ*-open set containing* x *is called* Δ*-neighborhood. We write* $\Delta N(x)$.

The proof of the following result is trivial and so it is omitted.

Proposition 3.7. *Let* (X,τ) *be a topological space and* $A \subseteq X$. *Then* $x \in \Delta Int(A)$, *if and only if, there is a* $\Delta N(x)$, *such that* $\Delta N(x) \subseteq A$.

Basic properties of Δ-interior are summarized in the next proposition.

Proposition 3.8. *Let* (X,τ) *be a topological space and* $A, B \subseteq X$. *Then*

(*i*) *If* $A \subseteq B$, then $\Delta Int(A) \subseteq \Delta Int(B)$.
(*ii*) $\Delta Int(A) \cup \Delta Int(B) \subseteq \Delta Int(A \cup B)$.
(*iii*) $\Delta Int(A \cap B) = \Delta Int(A) \cap \Delta Int(B)$.

Proof.

(*i*) Assume $A \subseteq B$ and given $x \in \Delta Int(A)$. Proposition 3.7 implies that there is a $\Delta N(x)$ satisfying $\Delta N(x) \subseteq A$. Being $A \subseteq B$, we get $\Delta N(x) \subseteq B$. Hence, $x \in \Delta Int(B)$.

(*ii*) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$ we get from part (*i*) that $\Delta Int(A) \subseteq \Delta Int(A \cup B)$ and $\Delta Int(B) \subseteq \Delta Int(A \cup B)$. Thus, $\Delta Int(A) \cup \Delta Int(B) \subseteq \Delta Int(A \cup B)$.

(*iii*) On one hand it is clear that $\Delta Int(A \cap B) \subseteq \Delta Int(A) \cap \Delta Int(B)$. On the other hand, let $x \in \Delta Int(A) \cap \Delta Int(B)$. Then there are Δ-neighborhoods of x, $\Delta N_1(x)$ and $\Delta N_2(x)$ such that $\Delta N_1(x) \subseteq A$ and $\Delta N_2(x) \subseteq B$. So, $\Delta N_1(x) \cap \Delta N_2(x) \subseteq A \cap B$. Therefore, $x \in \Delta Int(A \cap B)$.

In connection with Example 2.4 if $A = \{a, b\}$ and $B = \{c\}$, then $\Delta Int(A) = \Delta Int(B) = \phi$, however $\Delta Int(A \cup B) = A \cup B$. This shows that the inclusion in statement (*ii*) of Proposition 3.8 is strict.

Corollary 3.9. *Let* (X,τ) *be a topological space and* $A,B \subseteq X$. *Then*

(*i*) *If* $A \subseteq B$, then $\Delta Ext(B) \subseteq \Delta Ext(A)$.
(*ii*) $\Delta Ext(A) \cup \Delta Ext(B) \subseteq \Delta Ext(A \cap B)$.
(*iii*) $\Delta Ext(A \cup B) = \Delta Ext(A) \cap \Delta Ext(B)$.

Proposition 3.10. *Let* (X,τ) *be a* $T_{\frac{1}{2}}$*-space and* $A \subseteq X$. *Then* $\Delta Int(A) = A$.

Proof. It is known that $\Delta Int(A) \subseteq A$ by Lemma 3.3(*ii*). Conversely, let $x \in A$, then $\{x\}$ is a Δ-neighborhood of *x* by Proposition 2.6. So, $x \in \{x\} \subseteq A$. That is, $x \in \Delta Int(A)$. Therefore, $A \subseteq \Delta Int(A)$ and hence, $\Delta Int(A) = A$.

Corollary 3.11. *Let* (X,τ) *be a* $T_{\frac{1}{2}}$*-space and* $A,B \subseteq X$. *Then* $\Delta Int(A) \cup \Delta Int(B) = A \cup B = \Delta Int(A \cup B)$.

Example 2.8 confirms that the converse of Proposition 3.10 is not correct.

In the same manner we give a characterization of Δ-closure in terms of Δ-neighborhoods.

Proposition 3.12. *Let* (X,τ) *be a topological space and* $A \subseteq X$.*Then* $x \in \Delta Cl(A)$, *if and only if for any* $\Delta N(x)$, $\Delta N(x) \cap A \neq \phi$.

Proof. For the forward direction, assume $x \in \Delta Cl(A)$ but there is a $\Delta N(x)$, satisfying $\Delta N(x) \cap A = \phi$. So, $A \subseteq X - \Delta N(x)$. Hence, $X - \Delta N(x)$ is a Δ-closed set containing *A*. As $x \in \Delta Cl(A)$, we have $x \in X - \Delta N(x)$ which is absurd. For the backward direction, suppose that for any $\Delta N(x)$, Δ-neighborhood of *x*, $\Delta N(x) \cap A \neq \phi$, yet $x \notin \Delta Cl(A)$. Let $\mathcal{C} = \{C : C \text{ is } \Delta - closed, \text{ and } A \subseteq C\}$. Therefore,

$$x \notin \Delta Cl(A) \Leftrightarrow x \notin \bigcap \mathcal{C} \Leftrightarrow x \in \bigcup (X - \mathcal{C})$$

where, $X - \mathcal{C} = \{X - C : X - C \text{ is } \Delta - open, \text{ and } X - C \subseteq X - A\}$. So, there is a Δ-closed set C satisfying $x \in X - C \subseteq X - A$. That is $X - C$ is a Δ-neighborhood of *x* with $(X - C) \cap A = \phi$ which is a contradiction.

Fundamental properties of Δ-closure are given below.

Proposition 3.13. *Let* (X,τ) *be a topological space and* $A,B \subseteq X$. *Then*

(*i*) *If* $A \subseteq B$, *then* $\Delta Cl(A) \subseteq \Delta Cl(B)$.
(*ii*) $\Delta Cl(A) \cup \Delta Cl(B) = \Delta Cl(A \cup B)$.
(*iii*) $\Delta Cl(A \cap B) \subseteq \Delta Cl(A) \cap \Delta Cl(B)$.

Proof.

(*i*) Suppose $A \subseteq B$ and let $x \in \Delta Cl(A)$. Let $\Delta N(x)$ be a Δ-neighborhood of *x*. Then Proposition 3.12 asserts that $\Delta N(x) \cap A \neq \phi$. As $A \subseteq B$, we get $\Delta N(x) \cap B \neq \phi$. So, $x \in \Delta Cl(B)$.
(*ii*) On one hand, $A,B \subseteq A \cup B$, so it follows by part (*i*) that $\Delta Cl(A) \cup \Delta Cl(B) \subseteq \Delta Cl(A \cup B)$. On the other hand, assume $x \in \Delta Cl(A \cup B)$ but $x \notin \Delta Cl(A) \cup \Delta Cl(B)$. Then by Proposition 3.12 there are Δ-neighborhoods of *x*, $\Delta N_1(x)$ and $\Delta N_2(x)$ such that $\Delta N_1(x) \cap A = \phi$ and $\Delta N_2(x) \cap B = \phi$. Hence, $(\Delta N_1(x) \cap \Delta N_2(x)) \cap (A \cup B) = \phi$. This implies that $x \notin \Delta Cl(A \cup B)$ which contradicts the assumption. Therefore, $\Delta Cl(A \cup B) \subseteq \Delta Cl(A) \cup \Delta Cl(B)$.
(*iii*) The conclusion follows directly from the fact that $A \cap B \subseteq A$ and $A \cap B \subseteq B$, and applying part (*i*).

Let's consider Example 2.4. For $A = \{a,b\}$ and $B = \{c,d\}$, we have $A \cap B = \phi$ and so $\Delta Cl(A \cap B) = \phi$. However, $\Delta Cl(A) = \{a,b,c\}$, $\Delta Cl(B) = \{a,b,c,d\}$ and so $\Delta Cl(A) \cap \Delta Cl(B) = \{a,b,c\}$. This assures that in statement (*iii*) of Proposition 3.13 the inclusion is strict.

The next result points out that in a $T_{\frac{1}{2}}$-space, each set and its Δ-closure are the same.

Proposition 3.14. *Let* (X,τ) *be a* $T_{\frac{1}{2}}$*-space and* $A \subseteq X$. *Then* $\Delta Cl(A) = A$.

Proof. Lemma 3.4 (3) shows that $A \subseteq \Delta Cl(A)$. Conversely, let $x \in \Delta Cl(A)$. Proposition 3.12 implies that $\Delta N(x) \cap A \neq \phi$, for any Δ-neighborhood of *x*. Proposition 2.6 involves that in particular $\{x\}$ is a Δ-neighborhood of *x*. Therefore, $x \in A$ and $\Delta Cl(A) = A$.

Corollary 3.15. *Let* (X,τ) *be a* $T_{\frac{1}{2}}$*-space and* $A, B \subseteq X$. *Then* $\Delta Cl(A \cap B) = A \cap B = \Delta Cl(A) \cap \Delta Cl(B)$.

Example 2.8 shows that the converse of Proposition 3.14 is not correct.

4. Δ-limit points and Δ-boundary points

In this section we give analogues to *limit points* and *boundary points* notions in terms of Δ open sets.

Definition 4.1. *Let* (X,τ) *be a topological space and* $A \subseteq X$. *A point* $x \in X$ *is called Δ-limit point of A if for any* $\Delta N(x)$, *Δ-neighborhood of x we have* $(\Delta N(x) \cap A) - \{x\} \neq \phi$. *The set of all Δ-limit points of a set A is called the Δ-derived set of A, and will be denoted by* $\Delta Der(A)$.

In a similar manner we denote $Der(A)$ the *derived set* of a set *A*. Since every open set is Δ-open, it is apparent that $\Delta Der(A) \subseteq Der(A)$.

Example 4.2. *Let* $X = \{a,b,c,d\}$ *with a topology* $\tau = \{\phi, X, \{a,b\}\}$. *Let* $\tau_{\Delta o}$ *denote the collection of all Δ-open sets in X. Then*

$$\tau_{\Delta o} = \{\phi, X, \{a,b\}, \{c,d\}\}$$

For $A = \{a,c\}$, *we get* $Der(A) = \{b,c,d\}$. *However* $\Delta Der(A) = \{b,d\}$.

Proposition 4.3 *Let* (X,τ) *be a topological space and* $A, B \subseteq X$. *Then*

(*i*) *If* $A \subseteq B$, *then* $\Delta Der(A) \subseteq \Delta Der(B)$.
(*ii*) $\Delta Der(A) \cup \Delta Der(B) = \Delta Der(A \cup B)$.
(*iii*) $\Delta Der(A \cap B) \subseteq \Delta Der(A) \cap \Delta Der(B)$.

Proof.

(*i*) Assume $A \subseteq B$. Let $x \in \Delta Der(A)$ and $\Delta N(x)$ be any Δ-neighborhood of *x*. Then $(\Delta N(x) \cap A) - \{x\} \neq \phi$, but $A \subseteq B$, so $(\Delta N(x) \cap B) - \{x\} \neq \phi$. Hence, $x \in \Delta Der(B)$.

(*ii*) It is known that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. So it follows from part (*i*) that $\Delta Der(A) \subseteq \Delta Der(A \cup B)$ and $\Delta Der(B) \subseteq \Delta Der(A \cup B)$. Thus, $\Delta Der(A) \cup \Delta Der(B) \subseteq \Delta Der(A \cup B)$.

Conversely, assume that $x \in \Delta Der(A \cup B)$ but $x \notin \Delta Der(A) \cup \Delta Der(B)$. Hence, there is $\Delta N(x)$ a Δ-neighborhood of *x* such that $A \cap \Delta N(x) = \{x\}$ and $B \cap \Delta N(x) = \{x\}$. So, $(A \cup B) \cap \Delta N(x) = (A \cap \Delta N(x)) \cup (B \cap \Delta N(x)) = \{x\}$. Therefore, $x \notin \Delta Der(A \cup B)$ which is a contradiction.

(*iii*) It is known that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Applying part (*i*) we get $\Delta Der(A \cap B) \subseteq \Delta Der(A)$ and $\Delta Der(A \cap B) \subseteq \Delta Der(B)$. Therefore, $\Delta Der(A \cap B) \subseteq \Delta Der(A) \cap \Delta Der(B)$.

We provide an example that shows the inclusion in part (*iii*) of Proposition 4.3 is strict.

Example 4.4 *Let* $X = \{a,b,c,d,e\}$ *with a topology*

$$\tau = \{\phi, X, \{a,b,c\}, \{d\}, \{a,b,c,d\}\}.$$

Let $\tau_{\Delta o}$ *denote the collection of all Δ-open sets in X. Then*

$$\tau_{\Delta o} = \{\phi, X, \{a,b,c\}, \{d\}, \{a,b,c,d\}, \{a,b,c,e\}, \{d,e\}, \{e\}\}.$$

For $A = \{a,b\}$ *and* $B = \{c,d,e\}$, *we get* $\Delta Der(A) = \{a,b,c\}$ *and* $\Delta Der(B) = \{a,b\}$. *Hence,* $\Delta Der(A) \bigcap, \Delta Der(B) = \{a,b\}$. *However,* $A \cap B = \phi$ *and so* $\Delta Der(A \cap B) = \phi$.

Proposition 4.5. *Let (X,τ) be a $T_{\frac{1}{2}}$-space and $A \subseteq X$. Then $\Delta Der(A) = \phi$.*

Proof. It is obvious that $\Delta Der(\phi) = \phi$. So, assume without loss of generality that $A \neq \phi$. Let $x \in \Delta Der(A)$. Then Proposition 2.6 confirms that $\{x\}$ is a Δ-neighborhood of *x*. We take into account two cases:

(*i*) $x \in A$. Then $\{x\} \cap A - \{x\} = \{x\} - \{x\} = \phi$, which is a contradiction.
(*ii*) $x \notin A$. Then $\{x\} \cap A - \{x\} = \phi - \{x\} = \phi$, which is a contradiction.

Therefore, $\Delta Der(A) = \phi$.

The converse of the previous proposition does not hold. In Example 2.8 each set has no Δ-limit points, yet the space is not $T_{\frac{1}{2}}$.

Definition 4.6 *Let (X,τ) be a topological space and $A \subseteq X$. A point $x \in X$ is called Δ-boundary point of A if for any $\Delta N(x)$, Δ-neighborhood of x we have $\Delta N(x) \cap A \neq \phi$ and $\Delta N(x) \cap (X - A) \neq \phi$. The set of all semi Δ-boundary points of a set A is called the Δ-boundary of A, and will be denoted by $\Delta Bd(A)$.*

Theorem 4.7 *Let (X,τ) be a topological space and $A \subseteq X$. Then*

(*i*) $\Delta Bd(A) = \Delta Cl(A) \cap \Delta Cl(X - A)$.
(*ii*) *The $\Delta Int(A)$ and $\Delta Bd(A)$ are disjoint.*
(*iii*) $\Delta Cl(A) = \Delta Int(A) \cup \Delta Bd(A)$.
(*iv*) *If $A \subseteq X$, then $X = \Delta Int(A) \cup \Delta Bd(A) \cup \Delta Ext(A)$.*
(*v*) *If A is Δ-clopen (both Δ-open and Δ-closed), then $\Delta Bd(A) = \phi$.*
(*vi*) *If A is Δ-open, then $\Delta Bd(A) = \Delta Cl(A) - A$.*

Proof.

(*i*) Trivial.
(*ii*) If $x \in \Delta Int(A)$, then by Proposition 3.7, there is a $\Delta N(x)$ such that $\Delta N(x) \subseteq A$. As a consequence of Proposition 3.12, $x \notin Cl(X - A)$. So, $x \notin \Delta Bd(A)$. Therefore, $\Delta Int(A) \cap \Delta Bd(A) = \phi$.
(*iii*) Clearly, $\Delta Int(A) \subseteq \Delta Cl(A)$ and $\Delta Bd(A) \subseteq \Delta Cl(A)$, so $\Delta Int(A) \cup \Delta Bd(A) \subseteq \Delta Cl(A)$. On the other hand, let $x \in \Delta Cl(A)$. We have two cases:
  - Case 1: $x \in \Delta Int(A)$. Then $x \in \Delta Int(A) \cup \Delta Bd(A)$, and we are done.
  - Case 2: $x \notin \Delta Int(A)$. Let $\Delta N(x)$ be any Δ-neighborhood of *x*. As a result of Proposition 3.7, $\Delta N(x) \cap (X - A) \neq \phi$. Thus, $x \in \Delta Cl(X - A)$. Hence, $x \in \Delta Bd(A)$ and accordingly $x \in \Delta Int(A) \cup \Delta Bd(A)$.
(*iv*) Let $A \subseteq X$. It follows from Proposition 3.13 that

$$X = \Delta Cl(X) = \Delta Cl\big(A \cup (X - A)\big) = \Delta Cl(A) \cup \Delta Cl(X - A).$$

Part (*iii*) above assures that

$$X = \Delta Int(A) \cup \Delta Bd(A) \cup \Delta Int(X - A) \cup \Delta Bd(x - A).$$

However, $\Delta Bd(x - A) = \Delta Bd(A)$ and $\Delta Int(X - A) = \Delta Ext(A)$. Therefore,

$$X = \Delta Int(A) \cup \Delta Bd(A) \cup \Delta Ext(A).$$

Finally, statements (*v*) and (*vi*) follow directly from part (*i*).

It follows from Theorem 4.7(*i*) and Lemma 3.3(*iii*) that $\Delta Bd(A) \subseteq Bd(A)$, where $Bd(A)$ denotes the boundary of *A* in a topological space. In context to Example 2.4, if $A = \{a,b,c,e\}$, then $\Delta Bd(A) = \Delta Cl(A) - A = \phi$, yet *A* is Δ-closed but not Δ-open. This indicates that the converse of statements (*v*) and (*vi*) in Theorem 4.7 fails to hold in general.

Taking into consideration Proposition 3.14 and Theorem 4.7 the following result is evident.

Proposition 4.8 *Let (X,τ) be a $T_{\frac{1}{2}}$-space and $A \subseteq X$. Then, $\Delta Bd(A) = \phi$.*

The converse of the above proposition is not valid. In Example 2.8 each set has an empty Δ-boundary, even so the space is not $T_{\frac{1}{2}}$.

5. Δ-dense sets

Recall that a set A in a topological space X is called *dense* if $Cl(A) = X$. Similarly we have the following notion.

Definition 5.1. *Let (X,τ) be a topological space and $D \subseteq X$. Then D is said to be Δ-dense in X if $\Delta Cl(D) = X$.*

It follows directly from Lemma 3.3(*iii*), that if a set is Δ-dense, then it is dense in the usual sense. Nonetheless, the converse need not be true. In Example 2.4, the set $\{a,b,c,e\}$ is dense in the usual sense, even so it is not Δ-dense.

Proposition 5.2. *Let (X,τ) be a topological space and $D \subseteq X$. Then D is Δ-dense in X if and only if, $\Delta Int(X-D) = \phi$.*

Proof. A set D is Δ-dense in X if and only if $\Delta Cl(D) = X$ if and only if $X - \Delta Cl(D) = \phi$ if and only if $\Delta Int(X-D) = \phi$, as consequence of Proposition 3.4.

Proposition 5.3. *Let (X,τ) be a topological space and D be a Δ-dense set in X. If $A \subseteq X$, then $\Delta Int(A) \subseteq \Delta Cl(D \cap \Delta Int(A))$.*

Proof. Suppose D is Δ-dense. Let $x \in \Delta Int(A)$, then there is $\Delta M(x)$ a Δ-neighborhood of x so that $\Delta M(x) \subseteq A$. Let $\Delta N(x)$ be any Δ-neighborhood of x. Since D is Δ-dense, we get $\Delta N(x) \cap \Delta M(x) \cap D \neq \phi$. So, $\Delta N(x) \cap \Delta M(x) \cap D \cap \Delta Int(A) = \Delta N(x) \cap \Delta M(x) \cap D \neq \phi$. On account of $\Delta N(x) \cap \Delta M(x) \cap D \cap \Delta Int(A) \subseteq \Delta N(x) \cap D \cap \Delta Int(A)$, we have $\Delta N(x) \cap D \cap \Delta Int(A) \neq \phi$. Therefore, $x \in \Delta Cl(D \cap \Delta Int(A))$ and $\Delta Int(A) \subseteq \Delta Cl(D \cap \Delta Int(A))$.

Corollary 5.4. *Let (X,τ) be a topological space and D be a Δ-dense set in X. If O is Δ-open, then $O \subseteq \Delta Cl(D \cap O)$.*

The next result follows readily from Proposition 3.14.

Proposition 5.5. *Each proper subset of a $T_{\frac{1}{2}}$-space is not Δ-dense.*

Proof. Let A be a proper subset of $T_{\frac{1}{2}}$-space X. Proposition 3.14 guarantees that $\Delta Cl(A) = A \neq X$. Thus, A is not Δ-dense.

6. Δ-open sets and Δ-closed sets versus product topology

We recall the following lemma which is going to be utilized later.

Lemma 6.1. *Let A,B,C and D be sets. Then*

(*i*) $(A \cap C) \times (B \cap D) = (A \times B) \cap (C \times D)$.
(*ii*) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.
(*iii*) *If $A \subseteq X$ and $B \subseteq Y$, then*

$$X \times Y - (A \times B) = \left(X \times (Y - B)\right) \cup \left((X - A) \times B\right).$$

Proposition 6.2. *Let (X_1,τ_1) and (X_2,τ_2) be two topological spaces. If A_1 is Δ-open in X_1 and A_2 is Δ-open in X_2, then $A_1 \times A_2$ is Δ-open in the product topology $X_1 \times X_2$.*

Proof. Suppose A_1 is Δ-open in X_1 and A_2 is Δ-open in X_2. Then Theorem 2.2 assures that there are an open set O_1 and a closed set C_1 in X_1, and there are an open set O_2 and a closed set C_2 in X_2, such that $A_1 = O_1 \cap C_1$ and $A_2 = O_2 \cap C_2$. So Lemma 6.1(*i*) implies that

$$A_1 \times A_2 = (O_1 \cap C_1) \times (O_2 \cap C_2) = (O_1 \times O_2) \cap (C_1 \times C_2).$$

Since $O_1 \times O_2$ is open in $X_1 \times X_2$ and $C_1 \times C_2$ is closed in $X_1 \times X_2$ we get $A_1 \times A_2$ is Δ-open in $X_1 \times X_2$.

Corollary 6.3. *Let* (X_1, τ_1) *and* (X_2, τ_2) *be two topological spaces. If* B_1 *is Δ-closed in* X_1 *and* B_2 *is Δ-closed in* X_2*, then* $(X_1 \times B_2) \cup (B_1 \times X_2)$ *is Δ-closed in the product topology* $X_1 \times X_2$.

Proof. Suppose B_1 is Δ-closed in X_1 and B_2 is Δ-closed in X_2. Obviously, $X_1 - B_1$ is Δ-open in X_1 and $X_2 - B_2$ is Δ-open in X_2. As a result of Proposition 6.2 we have $(X_1 - B_1) \times (X_2 - B_2)$ is Δ-open in $X_1 \times X_2$. Thus, the complement of $(X_1 - B_1) \times (X_2 - B_2)$ is Δ-closed in $X_1 \times X_2$. As a consequence of Lemma 6.1 we get

$$X_1 \times X_2 - \left((X_1 - B_1) \times (X_2 - B_2)\right) = (X_1 \times B_2) \cup (B_1 \times X_2).$$

In other words, $(X_1 \times B_2) \cup (B_1 \times X_2)$ is Δ-closed in $X_1 \times X_2$.

Corollary 6.4. *Let* (X, τ) *be a topological space. If B is Δ-closed in X, then and* $(X \times B) \cup (B \times X)$ *is Δ-closed in the product topology* $X \times X$.

The converse of Proposition 6.2 need not be true. Also, the Cartesian product of two Δ-closed sets need not be Δ-closed in the product topology.

Example 6.5. *Let* $X = \{a, b, c, d\}$ *with a topology* $\tau = \{\phi, X, \{a, b\}, \{a\}\}$. *The collection of all Δ-open sets is* $\tau_{\Delta o} = \{\phi, X, \{a, b\}, \{a\}, \{c, d\}, \{c, b, d\}, \{b\}\}$. *The product topology on* $X \times X$ *is*

$$\tau_{X\times X} = \Big\{\phi, X \times X, \{(a,a),(b,a),(c,a),(d,a)\}, \{(a,a),(b,a)\}, \{(a,a),(a,b)\},$$
$$\{(a,a),(a,b),(b,a)\}, \{(a,a)\}, \{(a,a),(b,a),(c,a),(d,a),(a,b),(b,b),(c,b),(d,b)\}\Big\}.$$

The set $\{(a,a),(a,b),(b,a)\}$ *is open in* $X \times X$, *so it is Δ-open. However, there are no two Δ-open sets in X whose product equals* $\{(a,a),(a,b),(b,a)\}$.

Additionally, the set $\{a, c, d\}$ is Δ-closed in X, yet $\{a, c, d\} \times \{a, c, d\}$ is not Δ-closed in $X \times X$.

Let (X_1, τ_1) and (X_2, τ_2) be two topological spaces. Let B_1 be Δ-closed in X_1 and B_2 be Δ-closed in X_2. Then Corollary 2.3 guarantees that there are an open set O_1 and a closed set C_1 in X_1, and there are an open set O_2 and a closed set C_2 in X_2, such that $B_1 = O_1 \cup C_1$ and $B_2 = O_2 \cup C_2$.

Since $O_1 \times O_2$ is open in $X_1 \times X_2$ and $C_1 \times C_2$ is closed in $X_1 \times X_2$ we get $B = (O_1 \times O_2) \cup (C_1 \times C_2)$ is Δ-closed in $X_1 \times X_2$.

Due to Lemma 6.1(*ii*), we have $B \subseteq B_1 \times B_2$. In spite of the Cartesian product of two Δ-closed sets $B_1 = O_1 \cup C_1$ and $B_2 = O_2 \cup C_2$ need not be Δ-closed, it does contain a Δ-closed subset B in $X_1 \times X_2$ of the form $B = (O_1 \times O_2) \cup (C_1 \times C_2)$.

7. Δ-open sets in subspace topology

Let (X, τ) be a topological space and $Y \subseteq X$. Then it is known that the collection

$$\tau_Y = \{Y \cap U : U \text{ is open in } X\}$$

forms a topology on Y, named as the *subspace topology*. Recall that a set B is closed in Y if and only if there is a closed set C in X satisfying $B = Y \cap C$. We aim in this section to study Δ-open and Δ-closed sets in the subspace topology.

Proposition 7.1. *Let (X,τ) be a topological space and $Y \subseteq X$. Then, S is Δ-open in Y if and only if there is a Δ-open set A in X such that $S = Y \cap A$.*

Proof. Let $S \subseteq Y$. Then, S is Δ-open in Y if and only if there is an open set U in Y and a closed set K in Y such that $S = U \cap K$ if and only if there is an open set O in X and a closed set C in X such that $S = O \cap C \cap Y$. The conclusion follows by taking $A = O \cap C$ and employing Theorem 2.2.

Proposition 7.2. *Let (X,τ) be a topological space and $Y \subseteq X$. Then, S is Δ-closed in Y if and only if there is a Δ-closed set B in X such that $S = Y \cap B$.*

Proof. Let $S \subseteq Y$. Then, S is Δ-closed in Y if and only if there is an open set U in Y and a closed set K in Y such that $S = U \cup K$ if and only if there is an open set O in X and a closed set C in X such that $S = (Y \cap O) \cup (Y \cap C) = Y \cap (O \cup C)$. The conclusion follows by taking $B = O \cup C$ and applying Corollary 2.3.

Proposition 7.3. *Let (X,τ) be a topological space and $Y \subseteq X$. If S is Δ-open in Y and Y is Δ-open in X, then S is Δ-open in X.*

Proof. Since S is Δ-open in Y, $S = Y \cap A$ for some Δ-open set in X. Since Y and A are both Δ-open in X, so is $Y \cap A$.

Let (X,τ) be a topological space, $Y \subseteq X$, and $A \subseteq Y$. Let $\Delta Cl_Y(A)$ denote the Δ-closure of A in Y and $\Delta Int_Y(A)$ denote the Δ-interior of A in Y.

Proposition 7.4. *Let (X,τ) be a topological space, $Y \subseteq X$, and $A \subseteq Y$. Then $\Delta Cl_Y(A) \subseteq Y \cap \Delta Cl(A)$. Moreover, if $\Delta Cl_Y(A)$ is Δ-closed in Y, then $\Delta Cl_Y(A) = Y \cap \Delta Cl(A)$.*

Proof. Assume $Y \subseteq X$, and $A \subseteq Y$. Let

$$\mathcal{C}_X = \{C : C \text{ is } \Delta-\text{closed in } X, \text{and } A \subseteq C\}$$

and

$$\mathcal{C}_Y = \{C : C \text{ is } \Delta-\text{closed in } Y, \textit{and } A \subseteq C\}$$

Then

$$Y \cap \Delta Cl(A) = Y \cap \bigcap \Delta \mathcal{C}_X = \bigcap \Delta (Y \cap \mathcal{C}_X) \supseteq \bigcap \Delta \mathcal{C}_Y = \Delta Cl_Y(A)$$

where $Y \cap \mathcal{C}_X = \{Y \cap C : C \text{ is } \Delta-\text{closed in } X, \text{and } A \subseteq C\}$.

Next, suppose that $\Delta Cl_Y(A)$ is Δ-closed in Y. Then Proposition 7.2 implies that there is a Δ-closed set B in X such that $\Delta Cl_Y(A) = Y \cap B$. Clearly, $B \supseteq A$, so $\Delta Cl(A) \subseteq B$. Hence, $Y \cap \Delta Cl(A) \subseteq Y \cap B$. Therefore, $Y \cap \Delta Cl(A) \subseteq \Delta Cl_Y(A)$. Consequently, $\Delta Cl_Y(A) = Y \cap \Delta Cl(A)$.

Proposition 7.5. *Let (X,τ) be a topological space, $Y \subseteq X$, and $A \subseteq Y$. Then $\Delta Int(A) \subseteq \Delta Int_Y(A)$.*

Proof. Suppose $Y \subseteq X$, and $A \subseteq Y$. Let

$$\mathcal{O}_X = \{O : O \text{ is } \Delta-\text{open in } X, \text{and } O \subseteq A\}$$

and

$$\mathcal{O}_Y = \{O : O \text{ is } \Delta-\text{open in } Y, \text{and } O \subseteq A\}$$

Then

$$\Delta Int(A) = Y \cap \Delta Int(A) = Y \cap \bigcup \mathcal{O}_X = \bigcup (Y \cap \mathcal{O}_X) \subseteq \bigcup \mathcal{O}_Y = Int_Y(A)$$

where $Y \cap \mathcal{O}_X = \{Y \cap O : O \text{ is } \Delta-\text{open in } X, \text{and } O \subseteq A\}$.

Generally the inclusion in Proposition 7.5 is strict. With reference to Example 2.4, let $Y = \{a,b,d\}$ and $A = \{a,b\}$. Then $\Delta Int(A) = \phi$ but $\Delta Int_Y(A) = A$.

8. Conclusion

In this paper we provide characterizations for Δ-open sets and Δ-closed sets that are independent of the symmetric difference operation; Theorem 2.2 and Corollary 2.3. We define and investigate the notions of Δ-interior, Δ-closure, Δ-limit points, Δ-boundary points, and Δ-dense sets. The investigation continues to Δ-open and Δ-closed sets in product topology and in subspace topology. Many of the properties of those "Δ notions" agree with their corresponding topological notions. However, several counter examples are given to show distinction between the "Δ notions" and the usual topological notions.

For a future work we are going to define and investigate the concepts of Δ-*continuous functions* and Δ-*irresolute functions* in topological spaces.

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