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A novel iterative approach for split feasibility problem

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Abstract

The objective of this article is to study a three step iteration process in the framework of Banach spaces and obtain convergence results for generalized α-Reich-Suzuki nonexpansive mappings. We also provide numerical examples that support our main results and illustrate the convergence behaviour of the proposed process. In the end, we discuss about the solution of split feasibility problem by utilizing our results.

Keywords: Generalized α-Reich-Suzuki nonexpansive mappings, Fixed point, Strong and weak convergence, Split feasibility problem

AMS Subject Classification: 47H09, 47H10, 54H25

1. Introduction

Fixed point theorems are really important because of their applications. Banach fixed point result is the most famous fixed point theorem. It has only two conditions, underlying mapping must be contraction and involved space must be complete. Its wide applications to different fields of mathematics as well as out side mathematics are well known. Researchers generalized this great theorem in two ways: either by weakening the involved contraction condition or via generalizing the metric structure. Banach fixed point theorem is not satisfied by the arbitrary nonexpansive mapping which is very

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natural class of mappings. Hence in 1965, Kirk [1] proved a very basic existence results in respect of nonexpansive mappings. After this, many generalization of nonexpansive mappings came into picture. Generalizations due to Suzuki [2], Garcia et al. [3] and Aoyama and Kohsaka [4] are worth mentioning. It is worth mentioning that nonexpansive mappings are always continuous but geenralized nonexapnsive mappings need not be continuous in general. In 2017, Pant and Shukla [5] introduced the class of generalized α-nonexpansive mappings and established some existence and convergence theorems for the newly introduced class of mappings.

In 2018, Pandey et al. [6] introduced the class of generalized α-Reich-Suzuki nonexpansive mappings and obtained convergence theorems.

Anothor direction of fixed point theory is to construct iteration process to reach fixed points of nonlinear mappings. Mann iteration [7] and Ishikawa iteration [8] are one and two step process. In 2000, Noor studied the convergence criteria of the three-step iteration method for solving general variational inequalities and related problems. Glowinski and Le Tallec [9] employed three-step iterative approaches to find solutions for the problem of elastoviscoplasticity, eigenvalue computation and the theory of liquid crystals. In [9], it was shown that the three-step iterative process yields better numerical results than the estimated iterations in two and one steps. For concrete application of fixed point iteration process one can see [10–14]. Owing to importance of these study, many three step iteration due to Noor [15], Agarwal et al. iteration [16], Abbas and Nazir iteration [17], Thakur et al. iterations [18, 19], M* iteration [20], M iteration [21], K iteration [22] and K* iteration [23] came into picture.

Motivated by foregoing studies, Ullah et al. [24] and Temir and Korkut [25] introduced a new iteration involving generalized α-nonexpansive mappings. If S is a is a mapping on convex subset K of a Banach space E, then the process as follows:

$$\begin{cases} a_1 \in K, \\ c_n = \mathcal{S}((1-\phi_n)a_n + \phi_n \mathcal{S}a_n), \\ b_n = \mathcal{S}c_n, \\ a_{n+1} = \mathcal{S}((1-\psi_n)\mathcal{S}c_n + \psi_n \mathcal{S}b_n), \end{cases} \tag{1.1}$$

where $\{\phi_n\}$ and $\{\psi_n\}$ are sequences in (0,1). Authors showed that their process converges faster than the many known iterations.

In this paper, we prove some convergence results involving the iteration process (1.1) for generalized α-Reich-Suzuki nonexpansive mappings which is bigger class of mappings than the class of generalized α-nonexpansive mappings. Thus, our results are the genuine generalization of results of Ullah et al. [24] and Temir and Korkut [25]. Further, we construct a numerical example which justify the our findings over the the existing iteration processes. In the last, we provide the solution of split feasibility problem.

2. Preliminaries

For making our paper self contained, we collect some basic definitions and needed results.

Definition 1. *A Banach space E is said to be uniformly convex if for each $\varepsilon \in (0,2]$ there is a $\delta > 0$ such that for $a, b \in E$ with $\|a\| \le 1$, $\|b\| \le 1$ and $\|a-b\| > \varepsilon$, we have*

$$\left\|\frac{a+b}{2}\right\| < 1-\delta.$$

Definition 2. *A Banach space E is said to satisfy the Opial's condition if for any sequence $\{a_n\}$ in E which converges weakly to $a \in E$ i.e. $a_n \rightharpoonup a$ implies that*

$$\limsup_{n\to\infty} \|a_n - a\| < \limsup_{n\to\infty} \|a_n - b\|$$

for all $b \in E$ with $b \ne a$.

Examples of Banach spaces satisfying this condition are Hilbert spaces and all l^p spaces $(1 < p < \infty)$. On the other hand, $L^p[0,2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition.

A mapping $\mathcal{S} : K \to E$ is demiclosed at $b \in E$ if for each sequence $\{a_n\}$ in K and each $a \in E$, $a_n \rightharpoonup a$ and $\mathcal{S}a_n \to b$ imply that $a \in K$ and $\mathcal{S}a = b$.

Let K be a nonempty closed convex subset of a Banach E, and let $\{a_n\}$ be a bounded sequence in E. For $a \in E$ write:

$$r(a,\{a_n\}) = \limsup_{n\to\infty} d(a,a_n).$$

The asymptotic radius of $\{a_n\}$ relative to K is given by

$$r(\{a_n\}) = \inf\{r(a,a_n) : a \in K\}$$

and the asymptotic center $A(K,\{a_n\})$ of $\{a_n\}$ is defined as:

$$A(K,\{a_n\}) = \{a \in K : r(a,\{a_n\}) = r(K,\{a_n\})\}.$$

It is known that, in a uniformly convex Banach space, $A(K,\{a_n\})$ consists of exactly one point. The following lemma due to Schu [26] is very useful in our subsequent discussion.

Lemma 1. *Let E be a uniformly convex Banach space and* $\{t_n\}$ *be any sequence such that* $0 < p \leq t_n \leq q < 1$ *for some* $p,q \in \mathbb{R}$ *and for all* $n \geq 1$. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be any two sequences of E such that* $\limsup_{n\to\infty} \| a_n \| \leq r$, $\limsup_{n\to\infty} \| b_n \| \leq r$ *and* $\limsup_{n\to\infty} \| t_n a_n + (1-t_n)b_n \| = r$ *for some* $r \geq 0$. *Then,* $\lim_{n\to\infty} \| a_n - b_n \| = 0$.

Recently, Pandey et al. [6] introduced generalized α-Reich-Suzuki nonexpansive mapping which properly contains the Reich-Suzuki nonexpansive and generalized α-nonexpansive mappings. Definition runs as follows:

Definition 3. [6] *Let K be a nonempty subset of a Banach space X. A self map* $\mathcal{S} : K \to K$ *is said to be generalized* α*-Reich-Suzuki nonexpansive mapping if there exist an* $\alpha \in [0,1)$ *and for each* $x, y \in K$

$$\frac{1}{2} \| x - \mathcal{S}x \| \leq \| x - y \| \Rightarrow \| \mathcal{S}x - \mathcal{S}y \| \leq max\{P(x,y), Q(x,y)\}$$

where

$$P(x,y) = \alpha \| \mathcal{S}x - x \| + \alpha \| \mathcal{S}y - y \| + (1-2\alpha) \| x - y \|.$$

and

$$Q(x,y) = \alpha \| \mathcal{S}x - y \| + \alpha \| \mathcal{S}y - x \| + (1-2\alpha) \| x - y \|.$$

The following results are very important to get our results:

Lemma 2. [6] *Let K be a nonempty subset of a Banach space E and* $\mathcal{S} : K \to K$ *a generalized* α*-Reich-Suzuki nonexpansive mapping. Then,*

- $F(\mathcal{S})$ *is closed. Moreover, if E is strictly convex and K is convex, then* $F(\mathcal{S})$ *is convex.*
- *If* $F(\mathcal{S}) \neq \emptyset$, *then* $\mathcal{S}$ *is quasi-nonexpansive.*
- $\| a - \mathcal{S}b \| \leq \frac{3+\alpha}{1-\alpha} \| a - \mathcal{S}a \| + \| a - b \|$
  for all a and $b \in K$.

The above lemma shows that generalized α-Reich-Suzuki nonexpansive mapping satisfies condition (E) with $\mu = \left(\frac{3+\alpha}{1-\alpha}\right)$. Therefore generalized α-nonexpansive, Reich-Suzuki type nonexpansive and generalized α-Reich-Suzuki nonexpansive mapping belong to the class of mappings satisfying the condition (E).

Lemma 3. [19] *Let $\mathcal{S}$ be ageneralized α-Reich-Suzuki nonexpansive mapping defined on a nonempty closed subset K of a Banach space E with the Opial property. If a sequence $\{a_n\}$ converges weakly to c and $\lim_{n\to\infty}\| \mathcal{S}a_n - a_n \| = 0$, then $(I-\mathcal{S})$ is demiclosed at zero.*

3. Convergence Results

First, we prove few lemmas which will be useful in obtaining convergence results.

Lemma 4. *Let $\mathcal{S}$ be a generalized α-Reich-Suzuki nonexpansive mapping defined on a nonempty closed convex subset K of a Banach space E with $F(\mathcal{S}) \neq \emptyset$. Let $\{a_n\}$ be the iterative sequence defined by the iteration process* (1.1). *Then, $\lim_{n\to\infty}\| a_n - p \|$ exists for all $p \in F(\mathcal{S})$.*

Proof. Let $p \in F(\mathcal{S})$. In view of second part of Lemma 2, $\mathcal{S}$ is quasi-nonexpansive hence we have

$$\begin{aligned} \| c_n - p \| &= \| \mathcal{S}((1-\phi_n)a_n + \phi_n \mathcal{S}a_n) - p \| \\ &\leq \| (1-\phi_n)a_n + \phi_n \mathcal{S}a_n - p \| \\ &\leq (1-\phi_n)\| a_n - p \| + \phi_n \| \mathcal{S}a_n - p \| \\ &\leq (1-\phi_n)\| a_n - p \| + \phi_n \| a_n - p \| \\ &= \| a_n - p \| \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \| b_n - p \| &= \| \mathcal{S}c_n - p \| \\ &\leq \| a_n - p \|. \end{aligned} \tag{3.2}$$

Using (3.1) and (3.2), we get

$$\begin{aligned} \| a_{n+1} - p \| &= \| \mathcal{S}((1-\psi_n)\mathcal{S}c_n + \psi_n \mathcal{S}b_n) - p \| \\ &\leq \| (1-\psi_n)\mathcal{S}c_n + \psi_n \mathcal{S}b_n - p \| \\ &\leq (1-\psi_n)\| \mathcal{S}c_n - p \| + \psi_n \| \mathcal{S}b_n - p \| \\ &\leq \| c_n - p \| \\ &\leq \| a_n - p \|. \end{aligned} \tag{3.3}$$

Thus, $\{\| a_n - p \|\}$ is bounded and non-increasing sequence of reals and hence $\lim_{n\to\infty}\| a_n - p \|$ exists.

Lemma 5. *Let $\mathcal{S}$ be a generalized α-Reich-Suzuki nonexpansive mapping defined on a nonempty closed convex subset K of a uniformly convex Banach space E. Let $\{a_n\}$ be the iterative sequence defined by the iteration process* (1.1). *Then, $F(\mathcal{S}) \neq \emptyset$ if and only if $\{a_n\}$ is bounded and $\lim_{n\to\infty}\| \mathcal{S}a_n - a_n \| = 0$.*

Proof. Suppose $F(\mathcal{S}) \neq \emptyset$ and let $p \in F(\mathcal{S})$. Then, by Lemma 4, $\lim_{n\to\infty}\| a_n - p \|$ exists. Let

$$\lim_{n\to\infty}\| a_n - p \| = d. \tag{3.4}$$

From inequality (3.3), we have

$$\| a_{n+1} - p \| \leq \| c_n - p \| \leq \| a_n - p \|. \tag{3.5}$$

Owing to (3.4), we have

$$\lim_{n\to\infty}\| c_n - p \| = d. \tag{3.6}$$

Also, using the fact that $\mathcal{S}$ is quasi-nonexpansive we have $\| \mathcal{S}a_n - p \| \leq \| a_n - p \|$, which gives

$$\limsup_{n\to\infty}\| \mathcal{S}a_n - p \| \leq d. \tag{3.7}$$

From (3.1), we have

$$\|c_n - p\| \leq \|(1-\phi_n)a_n + \phi_n \mathcal{S}a_n - p\| \leq \|a_n - p\|$$

which on using (3.4) and (3.6) gives

$$\lim_{n\to\infty} \|(1-\phi_n)a_n + \phi_n \mathcal{S}a_n - p\| = d. \tag{3.8}$$

Using (3.4), (3.7), (3.8) and Lemma 1, we conclude that $\lim_{n\to\infty} \|\mathcal{S}a_n - a_n\| = 0$. Conversely, suppose that $\{a_n\}$ is bounded and $\lim_{n\to\infty} \|a_n - \mathcal{S}a_n\| = 0$. Let $p \in A(K, \{a_n\})$, we have

$$\begin{aligned} r(\mathcal{S}p, \{a_n\}) &= \limsup_{n\to\infty} \|a_n - \mathcal{S}p\| \\ &\leq \left(\frac{3+\alpha}{1-\alpha}\right) \limsup_{n\to\infty} \|\mathcal{S}a_n - a_n\| + \limsup_{n\to\infty} \|a_n - p\| \\ &= \limsup_{n\to\infty} \|a_n - p\| \\ &= r(p, \{a_n\}). \end{aligned}$$

This implies that $\mathcal{S}p \in A(K, \{a_n\})$. Since E is uniformly convex, $A(K, \{a_n\})$ is singleton, therefore we get $\mathcal{S}p = p$.

Theorem 1. *Let $\mathcal{S}$ be a generalized α-Reich-Suzuki nonexpansive mapping defined on a nonempty closed convex subset K of a Banach space E which satisfies the Opial's condition with $F(\mathcal{S}) \neq \emptyset$. If $\{a_n\}$ is the iterative sequence defined by the iteration process* (1.1)*, then $\{a_n\}$ converges weakly to a fixed point of $\mathcal{S}$.*

Proof. Let $p \in F(\mathcal{S})$. Then, from Lemma 4 $\lim_{n\to\infty} \|a_n - p\|$ exists. In order to show the weak convergence of the iteration process (1.1) to a fixed point of $\mathcal{S}$, we will prove that $\{a_n\}$ has a unique weak sub sequential limit in $F(\mathcal{S})$ For this, let $\{a_{n_j}\}$ and $\{a_{n_k}\}$ be two subsequences of $\{a_n\}$ which converges weakly to u and v respectively. By Lemma 4, we have $\lim_{n\to\infty} \|\mathcal{S}a_n - a_n\| = 0$ and using the Lemma 3, we have $I - \mathcal{S}$ is demiclosed at zero. So $u, v \in F(\mathcal{S})$.

Next, we show the uniqueness. Since $u, v \in F(\mathcal{S})$, so $\lim_{n\to\infty} \|a_n - u\|$ and $\lim_{n\to\infty} \|a_n - v\|$ exists. Let $u \neq v$. Then, by Opial's condition, we obtain

$$\begin{aligned} \lim_{n\to\infty} \|a_n - u\| &= \lim_{j\to\infty} \|a_{n_j} - u\| \\ &< \lim_{j\to\infty} \|a_{n_j} - v\| \\ &= \lim_{n\to\infty} \|a_n - v\| \\ &= \lim_{k\to\infty} \|a_{n_k} - v\| \\ &< \lim_{k\to\infty} \|a_{n_k} - u\| \\ &= \lim_{n\to\infty} \|a_n - u\| \end{aligned}$$

which is a contradiction, so $u = v$. Thus, $\{a_n\}$ converges weakly to a fixed point of $\mathcal{S}$.

Now, we establish some strong convergence results.

Theorem 2. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E and $\mathcal{S}: K \to K$ be a generalized α-Reich-Suzuki nonexpansive mapping with $F(\mathcal{S}) \neq \emptyset$. If $\{a_n\}$ is defined by the iteration process* (1.1)*, then $\{a_n\}$ converges strongly to a point of $F(\mathcal{S})$ if and only if $\liminf_{n\to\infty} d(a_n, F(\mathcal{S})) = 0$.*

Proof. If the sequence $\{a_n\}$ converges to a point $p \in F(\mathcal{S})$, then it is obvious that $\liminf_{n\to\infty} d(a_n, F(\mathcal{S})) = 0$.

For the converse part, assume that $\liminf_{n\to\infty} d(a_n, F(\mathcal{S})) = 0$. From Lemma 4, we have $\lim_{n\to\infty} \|a_n - p\|$ exists for all $p \in F(\mathcal{S})$, which gives

$$\|a_{n+1} - p\| \le \|a_n - p\| \text{ for any } p \in F(\mathcal{S})$$

which yields

$$d(a_{n+1}, F(\mathcal{S}) \le d(a_n, F(\mathcal{S})). \tag{3.9}$$

Thus, $\{d(a_n, F(\mathcal{S}))\}$ forms a non-increasing sequence which is bounded below by zero as well, so we get that $\lim_{n\to\infty} d(a_n, F(\mathcal{S}))$ exists. As, $\liminf_{n\to\infty} d(a_n, F(\mathcal{S})) = 0$ so $\lim_{n\to\infty} d(a_n, F(\mathcal{S})) = 0$.

Now, there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ and a sequence $\{u_j\}$ in $F(\mathcal{S})$ such that $\|a_{n_j} - u_j\| \le \frac{1}{2^j}$ for all $j \in \mathbb{N}$. From the proof of Lemma 4, we have

$$\|a_{n_{j+1}} - u_j\| \le \|a_{n_j} - u_j\| \le \frac{1}{2^j}.$$

Using triangle inequality, we get

$$\begin{aligned}\|u_{j+1} - u_j\| &\le \|u_{j+1} - a_{n_{j+1}}\| + \|a_{n_{j+1}} - u_j\| \\ &\le \frac{1}{2^{j+1}} + \frac{1}{2^j} \\ &\le \frac{1}{2^{j-1}} \\ &\to 0 \text{ as } j \to \infty.\end{aligned}$$

So, $\{u_j\}$ is a Cauchy sequence in $F(\mathcal{S})$. By Lemma 2, $F(\mathcal{S})$ is closed, so $\{u_j\}$ converges to some $u \in F(\mathcal{S})$.

Again, owing to triangle inequality, we have

$$\|a_{n_j} - u\| \le \|a_{n_j} - u_j\| + \|u_j - u\|.$$

Letting $j \to \infty$, we have $\{a_{n_j}\}$ converges strongly to $u \in F(\mathcal{S})$.

Since $\lim_{n\to\infty} \|a_n - u\|$ exists by Lemma 4, therefore $\{a_n\}$ converges strongly to $u \in F(\mathcal{S})$.

A mapping $\mathcal{S} : K \to K$ is said to satisfy the Condition (A) ([28]) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|a - \mathcal{S}a\| \ge f(d(a, F(\mathcal{S})))$ for all $a \in K$, where $d(a, F(\mathcal{S})) = inf\{\|a - p\| : p \in F(\mathcal{S})\}$.

Now, we present a strong convergence result using the Condition (A).

Theorem 3. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E and $\mathcal{S} : K \to K$ be a generalized α-Reich-Suzuki nonexpansive mapping with $F(\mathcal{S}) \ne \emptyset$. If $\mathcal{S}$ satisfies the Condition (A) and $\{a_n\}$ is defined by the iteration process* (1.1)*, then $\{a_n\}$ converges strongly to a point of $F(\mathcal{S})$*

Proof. From 3.9, $\lim_{n\to\infty} d(a_n, F(\mathcal{S}))$ exists.

Also, by Lemma 5 we have $\lim_{n\to\infty} \|a_n - \mathcal{S}a_n\| = 0$.

It follows from the Condition (A) that

$$\lim_{n\to\infty} f(d(a_n, F(\mathcal{S}))) \le \lim_{n\to\infty} \|a_n - \mathcal{S}a_n\| = 0,$$

that $\lim_{n\to\infty} f(d(a_n, F(\mathcal{S}))) = 0$.

Since f is a non decreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$, therefore $\lim_{n\to\infty} d(a_n, F(\mathcal{S})) = 0$.

By Theorem 2, the sequence $\{a_n\}$ converges strongly to a point of $F(\mathcal{S})$.

4. Numerical Example

In this section, we construct following example of a generalized α-Reich Suzuki nonexpansive mapping.

Example 4.1: Define a mapping $\mathcal{S}:[-1,1]\to[-1,1]$ by

$$\mathcal{S}x=\begin{cases}\dfrac{-x}{11} & x\in[-1,0)\\ -x & x\in[0,1]/\dfrac{1}{11}\\ 0 & x=\dfrac{1}{11}\end{cases}.$$

For $x=\frac{1}{11}$ and $y=1$, we have

$$\frac{1}{2}d(x,\mathcal{S}x)=\frac{1}{22}\le\frac{10}{11}=d(x,y),$$

but

$$d(\mathcal{S}x,\mathcal{S}y)=1>\frac{10}{11}=d(x,y).$$

Thus $\mathcal{S}$ does not satisfy condition (C). Now we show that $\mathcal{S}$ satisfies the condition (E). We consider different cases as follows:

(i) Let $x,y\in[-1,0)$, we have

$$\begin{aligned}d(x,\mathcal{S}y)&\le d(x,\mathcal{S}x)+d(\mathcal{S}x,\mathcal{S}y)\\&\le d(x,\mathcal{S}x)+\frac{1}{11}|y-x|\\&\le d(x,\mathcal{S}x)+d(x,y).\end{aligned}$$

(ii) Let $x,y\in[0,1]/\frac{1}{11}$,

$$\begin{aligned}d(x,\mathcal{S}y)&\le d(x,\mathcal{S}x)+d(\mathcal{S}x,\mathcal{S}y)\\&\le d(x,\mathcal{S}x)+d(x,y).\end{aligned}$$

(iii) Let $x\in[-1,0)$ and $y\in[0,1]/\frac{1}{11}$,

$$\begin{aligned}d(x,\mathcal{S}y)&=|x+y|\le|x|+|y|\\&\le\frac{12}{11}|x|+|x-y|\,(\text{as } x<0 \text{ and } y\ge 0)\\&=d(x,\mathcal{S}x)+d(x,y).\end{aligned}$$

(iv) Let $x\in[-1,0)$ and $y=\frac{1}{11}$,

$$\begin{aligned}d(x,\mathcal{S}y)&=|x|\le\frac{12}{11}|x|+\left|x-\frac{1}{11}\right|\\&=d(x,\mathcal{S}x)+d(x,y).\end{aligned}$$

(v) Let $x \in [0,1]/\frac{1}{11}$ and $y = \frac{1}{11}$

$$\begin{aligned} d(x, \mathcal{S}y) &= |x| \leq 2|x| + \left|x - \frac{1}{11}\right| \\ &= d(x, \mathcal{S}x) + d(x, y). \end{aligned}$$

Thus $\mathcal{S}$ satisfy the condition (E) with $\mu \geq 1$ and has a fixed point 0. Now, using above example, we will show that Iteration (1.1) converges faster than S-iteration, Abbas iteration, Thakur New iteration, M-iteration and K^*-iteration. Let $\alpha_n = 0.75$, $\beta_n = 0.45$, $\gamma_n = 0.15$ for all $n \in \mathbb{N}$ and $x_1 = 0.3$, then we get the following Table 1 of iteration values and graph.

5. Application

If C and Q are nonempty, closedand convex subsets of two real Hilbert spaces H_1 and H_2 respectively and $A : H_1 \to H_2$ is a bounded and linear operator. Then, the split feasibility problem (abbreviate SFP) is to find $a \in C$ such that

$$Aa \in Q. \tag{5.1}$$

If the solution set $\Omega = \{a \in C : Aa \in Q\} = C \cap A^{-1}Q$ of the SFP (5.1) is nonempty, then Ω is closed, convex and nonempty set. Censor and Elfving [29] solved the class of inverse problems by using SFP. In 2002, Byrne [30] introduced CQ-algorithm for solving the SFP. In this, the iterative step a_k is calculated as follows:

$$a_{k+1} = P_C[I - \gamma A^*(I - P_Q)A]a_k, k \geq 0, \tag{5.2}$$

where $0 < \gamma < \frac{2}{\|A\|^2}$, P_C and P_Q denote the projections onto sets C and Q, respectively and $A^* : H_2^* \to H_1^*$ is the adjoint of A.

Feng et al. [31] proved the following important result:

Lemma 6. *Let operator* $\mathcal{S} = P_C[I - \gamma A^*(I - P_Q)A]$, *where* $0 < \gamma < \frac{2}{\|A\|^2}$. *Then,* $\mathcal{S}$ *is a nonexpansive map.*

Any $a \in C$ is the solution of SFP if and only if it solves the following fixed point equation:

$$P_C[I - \gamma A^*(I - P_Q)A]a = a, a \in C.$$

So, the solution set Ω is equal to the fixed point set of $\mathcal{S}$, i.e, $F(\mathcal{S}) = \Omega = C \cap A^{-1}Q \neq \emptyset$. For details, one can refer [32, 33].

As $\mathcal{S} = P_C[I - \gamma A^*(I - P_Q)A]$ is a nonexpansive map and owing to Lemma 6, it generalized α-nonexpansive mapping for $\alpha = 0$. Hence by using Theorem 1, we get the following main result:

Theorem 4. *If* $\{a_n\}$ *is the sequence generated by the iterative algorithm* (1.1) *with* $\mathcal{S} = P_C[I - \gamma A^*(I - P_Q)A]$ *then,* $\{a_n\}$ *converges weakly to the solution of SFP (5.1).*

By using Theorem 2, we have the following convergence theorem:

Theorem 5. *If* $\{a_n\}$ *is the sequence generated by the iterative algorithm* (1.1) *with* $\mathcal{S} = P_C[I - \gamma A^*(I - P_Q)A]$ *then,* $\{a_n\}$ *converges strongly to the solution of SFP (5.1) if and only if* $\liminf_{n \to \infty} d(a_n, \Omega) = 0$.

Table 1: Comparison of Convergence of Different Iteration.

Step	S-iteration	Thakur New Iteration	M-iteration	K*-iteration	Iteration (1.1)
1.0	0.3	0.3	0.3	0.3	0.3
2.0	−0.09750000000000	0.00886363636363	−0.01363636363636	−0.00495867768595	0.00012396694214
3.0	0.00560020661157	0.00026188016528	−0.00022539444027	−0.00002049040366	0.00000005122600
4.0	−0.00182006714876	0.00000773736852	−0.00000372552793	−0.00000008467109	0.00000000002116
5.0	0.00010454104697	0.00000022860407	−0.00000006157897	−0.00000000034988	0.00000000000009
6.0	−0.00003397584026	0.00000000675421	−0.00000000101783	−0.00000000000144	0.00000000000000
7.0	0.00000195150487	0.00000000019955	−0.00000000001682	−0.00000000000006	0.00000000000000
8.0	−0.00000063423908	0.00000000000589	−0.00000000000027	0.00000000000000	0.00000000000000
9.0	0.00000003642943	0.00000000000017	−0.00000000000005	0.00000000000000	0.00000000000000
10.0	−0.00000001183956	0.00000000000005	0.00000000000000	0.00000000000000	0.00000000000000
11.0	0.00000000068004	0.00000000000000	0.00000000000000	0.00000000000000	0.00000000000000
12.0	−0.00000000022101	0.00000000000000	0.00000000000000	0.00000000000000	0.00000000000000

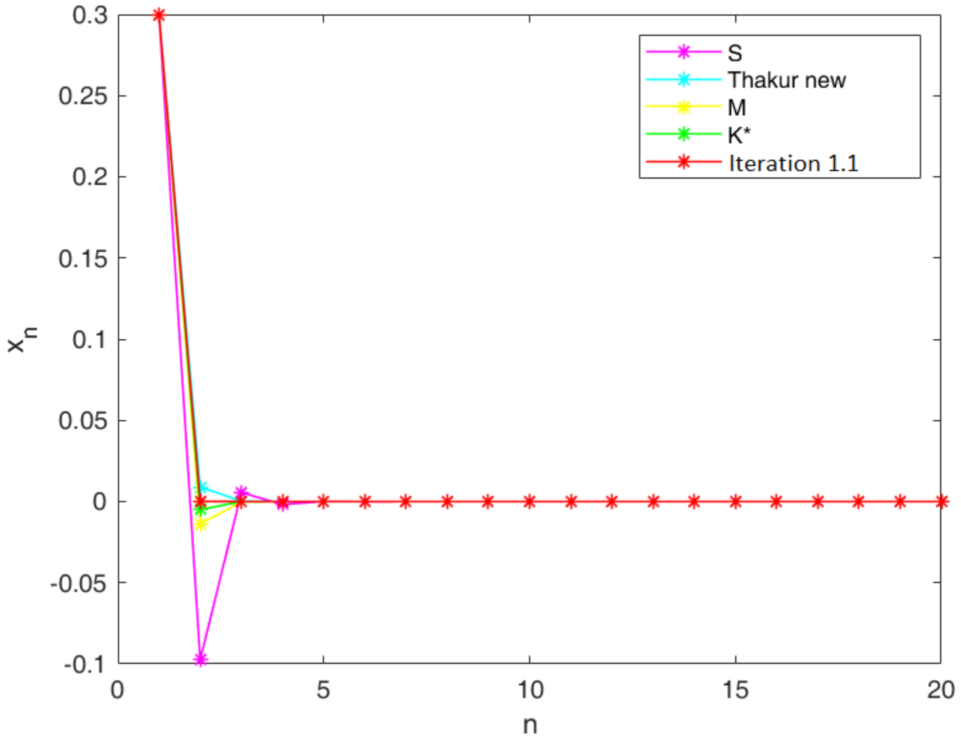

Figure 1: Graph corresponding to Table 1.

6. Conclusion

In this paper we studied the convergence behaviour of an iteration scheme introduced by Ullah et al. and Temir and Korkut [25] in respect of generalized α-Reich-Suzuki nonexpansive mapping. Theorems 1, 2 and 3 of our paper are main convergence theorems which generalized the Theorems 3.3 and 3.4 of Ullah et al. [24] and Theorem 2.2 and 2.3 of Temir and Korkut [25].

Conflict of interest

The authors declare that there is no confict of interests regarding the publication of this paper.

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