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Interpolation by Lupaş type operators on Tetrahedrons

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Abstract

The goal of this study is to build Lupaş type Bernstein operators (rational) on tetrahedrons with all straight edges and three curved edges determined by specific functions. Interpolation attributes, approximation accuracy (degree of exactness, precision set), and the remainders of the approximation formula of Lupaş type Bernstein operators are assessed using Peano's theorem and modulus of continuity.

Keywords: Lupaş type q-Bernstein operators, boolean sum operators, tetrahedron, modulus of continuity, error estimation.

MSC 2000: 41A35, 41A36, 41A80.

1. Introduction

Approximation is basically replacement of some thing complicated with the easier one. Approximation of functions for its use in computer calculations, a member of a given set, or some data has always attracted Mathematicians as it links both theoretical and applied Mathematics. Any development can be used in many industrial and commercial fields and thus requires further development in the subject to overcome challenges. Construction of Bernstein polynomials in 1912 is one of the elegent proof of the Weierstrass approximation theorem [3, 17] by S.N. Bernstein. In Computer-aided geometric

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design (CAGD), shape of the curves and surfaces are mimiced using the basis of Bernstein type polynomials [20].

Lupaş in 1987 [25] and Phillips [28] in 1997 respectively constructed the q-analogue of Bernstein polynomials [3] which are rational and polynomial respectively. A survey of the obtained results and references on the subject can be found in [27]. For results related to Lupaş, one can refer cf. [21, Chapter~10].

Approximating operators of Bernstein type on the tetrahedrons have uses in Computer-aided geometric design [23] and finite element analysis. For some recent literatures related to Bernstein type operators (see): λ-Bernstein operators and its approximation properties by Q. B. Cai et al. in [5, 6], N. Braha et al. studied convergence properties of λ-Bernstein operators via power series summability methods [14], Two dimentional Bernstein type operators and Kantorovich modifications by T. Acar in [12, 13]. Mursaleen et al studied error estimation for q-Bernstein shifted operators and generalized q-Bernstein Schurer operators in [8, 9]. One can refer [1, 14] for details related to Bernstein and its bivariate form with applications in CAGD. The blending interpolation operators were studied by Barnhill et al. in [22, 23, 24]. For interpolation on triangles and error bound, one can refer [7, 26]. Schumaker studied fitting surfaces to scattered data in [29]. For results related to Phillips and Lupaş type Bernstein operators on triangles, one can see recent work [10, 11]. Approximation properties for Bernstein type polynomials and its remainder terms are evaluated by D. D Stancu in [30, 31], R. Paltanea studied Durrmeyer type operators on a simplex in [16], A. Kajla and T. Acar studied blending type approximation by Bernstein durrmeyer type operators and α-Bernstein operators in [18, 19].

In [4], authors studied Bernstein-type Operators on Tetrahedrons. Motivated by their work, in this paper, our purpose is to construct Lupaş type q-Bernstein operators on tetrahedrons with all straight edges, on tetrahedrons with three curved edges defined by some functions and to study approximation properties. Classical Bernstein polynomials are used in [4], whereas we have used generalisation of Lupaş type q-Bernstein operators (rational). In case $q = 1$, it reduces to [4], thus due to presence of extra parameter q it has more flexibility in comparison to classical Bernstein polynomials.

2. Lupaş Type Operators on Straight Edges Tetrahedrons

Consider the tetrahedron H_d with vertices $V_0 = (0,0,0), V_1 = (h,0,0), V_2 = (0,h,0)$ and $V_3 = (0,0,h)$, with three edges τ_1, τ_2, τ_3 along the coordinate axes and with the edges $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ (opposite to the vertex V_0). Also, one denotes by $\sigma_{012}, \sigma_{013}, \sigma_{023}$ and σ_{123} the tetrahedron faces from the planes $V_0V_1V_2, V_0V_1V_3, V_0V_2V_3$ and $V_1V_2V_3$ respectively (see the left side of Figure 1). Let $T_i, i = 1,2,3$, be the triangles in which the planes $P_i, i = 1,2,3$, intersect the tetrahedron faces respectively in three points on the edges of tetrahedron as depicted in Figure 1.

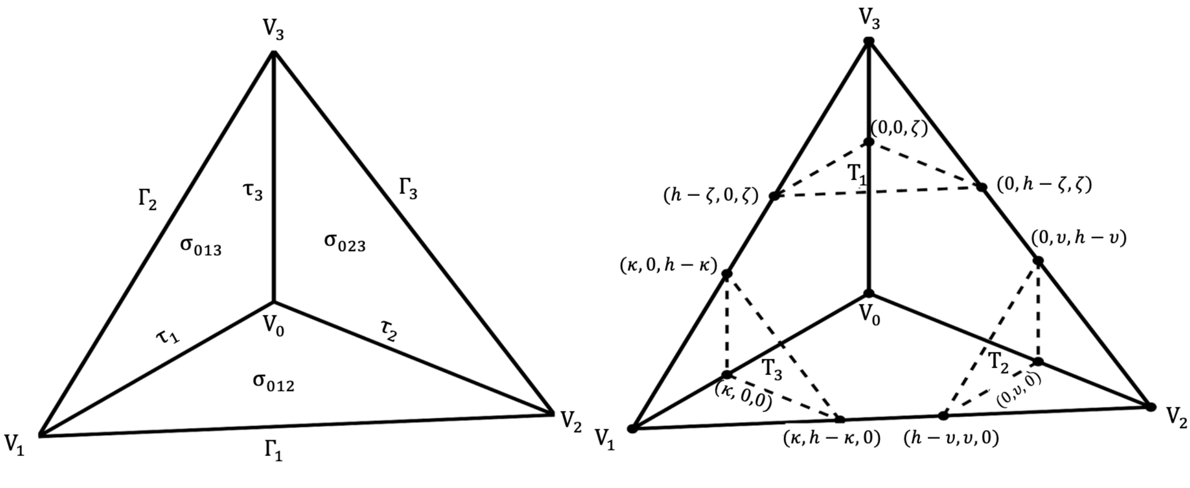

Figure 1: Tetrahedron with straight edges [4]

We will consider the triangle T_1 (as shown in Figure 2) to construct and study Lupaş Bernstein-type operators. Similar results can be obtained for the triangles T_2 and T_3.

2.1. Univariate Operators

For uniform partitions

$$\Delta^{\kappa}_{m,q}=\left\{\left(\frac{[i]_q}{[m]_l}(h-\upsilon-\zeta),\upsilon,\zeta\right)\mid i=\overline{0,m}\right\}\text{and}$$

$$\Delta^{\upsilon}_{n,q}=\left\{\left(\kappa,\frac{[j]_2}{[n]_q}(h-\kappa-\zeta),\zeta\right)\mid j=\overline{0,n}\right\},$$

the two Lupaş type Bernstein operators defined on each triangle are as

$$(B^{\kappa\upsilon}_{m,q}F)(\kappa,\upsilon,\zeta)=\sum_{i=0}^{m}p_{m,i}(\kappa,\upsilon,\zeta)F\left(\frac{[i]_q}{[m]_q}(h-\upsilon-\zeta),\upsilon,\zeta\right)$$

$$\text{and }(B^{\upsilon\kappa}_{n,\zeta}F)(\kappa,\upsilon,\zeta)=\sum_{j=0}^{n}q_{n,j}(\kappa,\upsilon,\zeta)F\left(\kappa,\frac{[j]}{[n]}(h-\kappa-\zeta),\zeta\right)$$

$$\text{where } q_{n,j}(\kappa,\upsilon,\zeta)=\frac{\begin{bmatrix} n\\ j\end{bmatrix}_i^{\frac{j(j-1)}{2}}\upsilon^j(h-\kappa-\zeta-\upsilon)^{m-j}}{\prod_{s=0}^{n-1}(h-\kappa-\zeta+q^s\upsilon)}.$$

For details related to the approximation accuracy (degree of exactness, precision set), one can refer [4]. The precision set (pres(P)) and the degree of exactness (dex(P)) will yield the order of an approximation operator P.

Theorem 2.1. If F: $T_h\to\mathbb{R}$, then

i. $B^{\kappa\upsilon}_{m,q}F=F$ on $\sigma_{023}\cup\sigma_{123}$,

$B^{\upsilon\kappa}_{n,q}F=F$ on $\sigma_{013}\cup\sigma_{123}$

ii. $\text{de}\kappa(B^{\kappa\upsilon}_{m,q})=\text{de}X(B^{\upsilon\kappa}_{n,q})=1$.

iii. $\text{pres }(B^{\kappa\upsilon}_{m,q})=\{\kappa^i\upsilon^j\zeta^k\mid i=0,1;j,k\in\mathbb{N}\}$

$\text{pres }(B^{j\kappa}_{n,q})=\{\kappa^i\upsilon^j\zeta^k\mid j=0,1;i,k\in\mathbb{N}\}$

iv. $$(B^{\kappa\upsilon}_{m,q}e_{2jk})(\kappa,\upsilon,\zeta)=\left[\kappa^2+\frac{\kappa(h-\upsilon-\zeta-\kappa)}{[m]_q}-\frac{\kappa^2(h-\kappa-\upsilon-\zeta)(1-q)}{(h-\upsilon-\zeta+q\kappa)}\left(1-\frac{1}{[m]_q}\right)\right]\upsilon^j\zeta^k.$$

$$(B^{\upsilon\kappa}_{n,q}e_{i2k})(\kappa,\upsilon,\zeta)=\left[\upsilon^2+\frac{\upsilon(h-\kappa-\upsilon-\zeta)}{[n]_q}-\frac{\kappa^2(h-\kappa-\upsilon-\zeta)}{(h-\kappa-\zeta+q\upsilon)}\left(1-\frac{1}{[n]_q}\right)\right]\kappa^i\zeta^k$$

$i,j,k\in\mathbb{N}$.

Proof. The relations

$$p_{m,i}(0,\upsilon,\zeta)=\begin{cases}1, & \text{for } i=0\\ 0, & \text{for } i>0\end{cases}$$

$$p_{m,i}(h-\upsilon-\zeta,\upsilon,\zeta)=\begin{cases}1, & \text{for } i=m\\ 0, & \text{for } i<m\end{cases}$$

respectively

$$q_{n,j}(\kappa,0,\zeta)=\begin{cases}1, & \text{for } j=0\\ 0, & \text{for } j>0\end{cases}$$

$$q_{n,j}(\kappa,h-\kappa-\zeta,\zeta)=\begin{cases}1, & \text{for } j=n\\ 0 & \text{for } j<n\end{cases}$$

imply that

$$(B^{\kappa\upsilon}_{m,q}F)(0,\upsilon,\zeta)=F(0,\upsilon,\zeta)$$
$$(B^{\kappa\upsilon}_{m,q}F)(h-\upsilon-\zeta,\upsilon,\zeta)=F(h-\upsilon-\zeta,\upsilon,\zeta)$$

and

$$(B^{\upsilon\kappa}_{n,q}F)(\kappa,0,\zeta)=F(\kappa,0,\zeta)$$
$$(B^{\upsilon\kappa}_{n,q}F)(\kappa,h-\kappa-\zeta,\zeta)=F(\kappa,h-\kappa-\zeta,\zeta)$$

Regarding the property (ii), we have

$$\begin{aligned}(B^{\kappa\upsilon}_{m,q}e_{000})(\kappa,\upsilon,\zeta) &=\sum_{i=0}^{m}p_{m,i}(\kappa,\upsilon,\zeta)\\ &=\frac{\sum_{i=0}^{m}\begin{bmatrix}m\\ i\end{bmatrix}_q q^{\frac{i(i-1)}{2}}\kappa^i(h-\kappa-\upsilon-\zeta)^{m-i}}{\prod_{r=0}^{m-1}(h-\upsilon-\zeta+q^r\kappa)}\\ &=\frac{\prod_{r=0}^{m-1}(h-\upsilon-\zeta+q^r\kappa)}{\prod_{r=0}^{m-1}(h-\upsilon-\zeta+q^r\kappa)}=1\end{aligned}$$

$$\begin{aligned}(B^{\kappa\upsilon}_{m,q}e_{(100)})(\kappa,\upsilon,\zeta) &=\frac{\sum_{i=0}^{m}\begin{bmatrix}m\\ i\end{bmatrix}_q q^{\frac{i(i-1)}{2}}\kappa^i(h-\kappa-\upsilon-\zeta)^{m-i}\frac{[i]_q}{[m]_2}(h-\upsilon-\zeta)}{\prod_{r=0}^{m-1}(h-\upsilon-\zeta+q^r\kappa)}\\ &=\frac{\sum_{i=0}^{m}\frac{[i]_q}{[m]_q}\begin{bmatrix}m\\ i\end{bmatrix}_q q^{\frac{i(i-1)}{2}}\kappa^i(h-\upsilon-\zeta-\kappa)^{m-i}}{\prod_{r=1}^{m-1}(h-\upsilon-\zeta+q^r\kappa)}\\ &=\frac{\sum_{i=0}^{m-1}\begin{bmatrix}m-1\\ i\end{bmatrix}_2 q^{\frac{i(i-1)}{2}}\kappa^{i+1}(h-\upsilon-\zeta-\kappa)^{m-i-1}}{\prod_{r=1}^{m-1}(h-\upsilon-\zeta+q^r\kappa)}\\ &=\frac{\kappa\sum_{i=0}^{m-1}\begin{bmatrix}m-1\\ i\end{bmatrix}_q q^{\frac{i(i-1)}{2}}(q\kappa)^i(h-\upsilon-\zeta-\kappa)^{m-i-1}}{\prod_{r=0}^{m-2}(h-\upsilon-\zeta+q^r(q\kappa)a)}\\ &=\kappa\end{aligned}$$

$$
\begin{aligned}
(B^{\kappa\upsilon}_{m,q}e_{200})(\kappa,\upsilon,\zeta) &= \frac{\sum_{i=0}^{m}\begin{bmatrix} m \\ i \end{bmatrix}_q q^{\frac{i(i-1)}{2}}\kappa^i(h-\upsilon-\zeta-\kappa)^{m-i}[i]_{q^2}(h-\upsilon-\zeta)^2}{\prod_{r=0}^{m-1}(h-\upsilon-\zeta-\kappa+q^r\kappa)[m]_{q^2}} \\
&= \frac{(h-\upsilon-\zeta)^2\sum_{i=0}^{m-1}\frac{[i+1]_q}{[m]_q}\begin{bmatrix} m-1 \\ i \end{bmatrix}_q q^{\frac{i(i-1)}{2}}\kappa^{i+1}(h-\upsilon-\zeta-\kappa)^{m-i-1}}{\prod_{r=0}^{m-1}(h-\upsilon-\zeta-\kappa+q^r\kappa)} \\
&= \frac{(h-\upsilon-\zeta)^2\kappa\sum_{i=0}^{m-1}\frac{1+q[i]_q}{[m]_q}\begin{bmatrix} m-1 \\ i \end{bmatrix}_q q^{\frac{i(i-1)}{2}}(q\kappa)^i(h-\upsilon-\zeta-\kappa)^{m-1-i}}{\prod_{r=0}^{m-1}(h-\upsilon-\zeta-\kappa+q^r\kappa)} \\
&= \frac{(h-\upsilon-\zeta)\frac{\kappa}{[m]_q}\sum_{i=0}^{m-1}\begin{bmatrix} m-1 \\ i \end{bmatrix}_q q^{\frac{i(i-1)}{2}}(q\kappa)^i(h-\upsilon-\zeta-\kappa)^{m-1-i}}{\prod_{r=0}^{m-2}(h-\upsilon-\zeta-\kappa+q^r(q\kappa))} \\
&\quad + \frac{(h-\upsilon-\zeta)^2\kappa\sum_{i=0}^{m-1}\frac{q[m-1]_q}{[m]_q}\frac{[i]_q}{[m-1]_q}\begin{bmatrix} m-1 \\ i \end{bmatrix}_q q^{\frac{i(i-1)}{2}}(q\kappa)^i(h-\upsilon-\zeta-\kappa)^{m-1-i}}{\prod_{r=0}^{m-2}(h-\upsilon-\zeta-\kappa+q^r(q\kappa))} \\
&= \left[(h-\upsilon-\zeta)\frac{\kappa}{[m]_q}+\frac{(h-\upsilon-\zeta)\kappa}{h-\upsilon-\zeta+q\kappa}\frac{q[m-1]_q}{[m]_q}\right]\cdot\frac{\sum_{i=0}^{m-2}\begin{bmatrix} m-2 \\ i \end{bmatrix}_q q^{\frac{i(i-1)}{2}}(q\kappa)^{i+1}(h-\upsilon-\zeta-\kappa)^{m-2-i}}{\prod_{r=0}^{m-3}(h-\upsilon-\zeta-\kappa+q^r(q^2\kappa))} \\
&= (h-\upsilon-\zeta)\frac{\kappa}{[m]_q}+\frac{(h-\upsilon-\zeta)q\kappa^2}{(h-\upsilon-\zeta+q\kappa)}\left(1-\frac{1}{[m]_q}\right)
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
(B^{\kappa\upsilon}_{m,q}e_{200})(\kappa,\upsilon,\zeta) &= \kappa^2\left(1-\frac{1}{[m]_q}\right)+(h-\upsilon-\zeta)\frac{\kappa}{[m]_q}-\left(\kappa^2-\frac{(h-\upsilon-\zeta)q\kappa^2}{h-\upsilon-\zeta+q\kappa}\left(1-\frac{1}{[m]_q}\right)\right) \\
&= \kappa^2+\frac{\kappa(h-\upsilon-\zeta-\kappa)}{[m]_q}-\frac{\kappa^2(h-\upsilon-\zeta-\kappa)}{(h-\upsilon-\zeta+q\kappa)}(1-q)\cdot\left(1-\frac{1}{[m]_q}\right)
\end{aligned}
$$

Remark 1. Additionally, it is demonstrable that

$$
\begin{aligned}
(B^{\upsilon\kappa}_{n,q}e_{000})(\kappa,\upsilon,\zeta) &= 1 \\
(B^{\upsilon\kappa}_{n,q}e_{0,0})(\kappa,\upsilon,\zeta) &= \upsilon \\
(B^{\upsilon\kappa}_{n,q}e_{i,2k})(\kappa,\upsilon,\zeta) &= \upsilon^2+\frac{\upsilon(h-\upsilon-\zeta-\kappa)}{[n]_q}-\frac{\upsilon^2(h-\upsilon-\zeta-\kappa)}{(h-\upsilon-\zeta-\kappa+q\upsilon)}\cdot\left(1-\frac{1}{[n]_q}\right) \\
(B^{\upsilon\kappa}_{n,q}e_{ijk})(\kappa,\upsilon,\zeta) &= \kappa^i\zeta^k(B^{\upsilon\kappa}_{n,q}e_{0j0})(\kappa,\upsilon,\zeta); \quad j=0,1,2; i,k\in\mathbb{N}.
\end{aligned}
$$

Theorem 2.2. If $F(\cdot,\upsilon,\zeta)\in C[0,h-\upsilon-\zeta]$, then for $\upsilon+\zeta\le h$

$$\left|(R_{m,q}^{\kappa\upsilon}F)(\kappa,\upsilon,\zeta)\right|\le 1+\frac{1}{\delta}\sqrt{\frac{\kappa(h-\upsilon-\zeta-\kappa)}{[m]_q}-\frac{\kappa^2(h-\upsilon-\zeta-\kappa)(1-q)}{(h-\upsilon-\zeta-\kappa+q\kappa)}\left(1-\frac{1}{[m]_q}\right)}W(F(\cdot,\upsilon,\zeta);\delta),$$

If $0<q\leqslant 1$, then

$$\left|(R_{m,q}^{\kappa\upsilon}F)(\kappa,\upsilon,\zeta)\right|\leqslant\left(1+\frac{h}{2\delta\sqrt{[m]_q}}\right)\cdot W(F(\cdot,\upsilon,\zeta);\delta)$$

Further, if $\delta=\frac{1}{\sqrt{[m]_q}}$, then

$$\left|(R_{m,q}^{\kappa\upsilon}F)(\kappa,\upsilon,\zeta)\right|\le\left(1+\frac{h}{2}\right)\cdot W\left(F(\cdot,\upsilon,\zeta);\frac{1}{\sqrt{[m]_q}}\right)$$

Proof. $\left|(R_{m,q}^{\kappa\upsilon}F)(\kappa,\upsilon,\zeta)\right|\le\sum_{i=0}^{m}p_{m,i}(\kappa,\upsilon,\zeta)F(\kappa,\upsilon,\zeta)-F\left(\frac{[i]_q}{[m]_q}(h-\upsilon-\zeta),\upsilon,\zeta\right)$, Since

$$\left|F(\kappa,\upsilon,\zeta)-F\left(\frac{[i]_q}{[m]_q}(h-\upsilon-\zeta),\upsilon,\zeta\right)\right|\le\left(\frac{1}{\delta}\left|\kappa-\frac{[i]_q}{[m]_q}(h-\upsilon-\zeta)\right|+1\right)W(F(\cdot,\upsilon,\zeta);\delta)$$

one obtains

$$\begin{aligned}\left|(R_{m,q}^{\kappa\upsilon}F)(\kappa,\upsilon,\zeta)\right|&\le\sum_{i=0}^{m}p_{m,i}(\kappa,\upsilon,\zeta)\left(\frac{1}{\delta}\left|\kappa-\frac{[i]_q}{[m]_q}(h-\upsilon-\zeta)\right|+1\right)W(F(\cdot,\upsilon,\zeta);\delta)\\&\le 1+\frac{1}{\delta}\sum_{i=0}^{m}\cdot p_{m,i}(\kappa,\upsilon,\zeta)\left(\kappa-\frac{[i]_q}{[m]_q}(h-\upsilon-\zeta)^2\right)^{1/2}W(F(\cdot,\upsilon,\zeta);\delta)\\&=1+\frac{1}{\delta}\cdot\sqrt{\frac{\kappa(h-\upsilon-\zeta-\kappa)}{[m]_q}-\frac{\kappa^2(h-\upsilon-\zeta-\kappa)}{(h-\upsilon-\zeta-\kappa+q\kappa)}\left(1-\frac{1}{[m]_q}\right)}W(F(\cdot,\upsilon,\zeta);\delta)\end{aligned}$$

If $0<q\le 1$, then we have

$$\frac{\kappa(h-\upsilon-\zeta-\kappa)}{[m]_q}\ge\frac{\kappa^2(h-\upsilon-\zeta-\kappa)(1-q)}{h-\upsilon-\zeta-\kappa+q\kappa}\cdot\left(1-\frac{1}{[m]_q}\right)$$

Finally, we have

$$\left|(R_{m,q}^{\kappa\upsilon}F)(\kappa,\upsilon,\zeta)\right|\leqslant\left[1+\frac{1}{\delta}\sqrt{\frac{\kappa(h-\upsilon-\zeta-\kappa)}{[m]_q}}\right]W(F(\cdot,\upsilon,\zeta);\delta)$$

Since $\max_{T_1}|\kappa(h-\upsilon-\zeta-\kappa)|=\frac{h^2}{4}$. It follows that

$$\left|R_{m,q}^{\kappa\upsilon}F(\kappa,\upsilon,\zeta)\right|\leqslant\left(1+\frac{h}{2\delta\sqrt{[m]_q}}\right)\cdot W(F(\cdot,\upsilon,\zeta);\delta)$$

For $\delta = \frac{1}{\sqrt{[m]_q}}$, we obtain

$$\left|(R^{\kappa\upsilon}_{m,q}F)(\kappa,\upsilon,\zeta)\right| \leqslant \left(1+\frac{h}{2}\right)W\left(F(\cdot,\upsilon,\zeta);\frac{1}{\sqrt{[m]_q}}\right)$$

We also have

$$\left|(R^{\upsilon\kappa}_{n,q}F)(\kappa,\upsilon,\zeta)\right| \leqslant \left(1+\frac{h}{2}\right)W\left(F(\kappa,\cdot,\zeta);\frac{1}{\sqrt{[n]_q}}\right)$$

Theorem 2.3. If $F(\cdot,\upsilon,\zeta)\in C^2[0,h]$, then for $0\le\xi\le h-\upsilon-\zeta\ \ \upsilon,\zeta\in[0,h]$, we have

$$(R^{\kappa\upsilon}_{m,q}F)(\kappa,\upsilon,\zeta) = -\left(\frac{\kappa(h-\upsilon-\zeta-\kappa)}{2[m]_q} - \frac{\kappa^2(h-\upsilon-\zeta-\kappa)(1-q)}{2(h-\upsilon-\zeta-\kappa+q\kappa)}\cdot\left(1-\frac{1}{[m]_q}\right)\right)F^{(2,0,0)}(\xi,\upsilon,\zeta)$$

and if $0<q\le 1$

$$\left|(R^{\kappa\upsilon}_{m,q}F)(\kappa,\upsilon,\zeta)\right| \le \frac{h^2}{8[m]_q}M_{200}F$$

where $M_{ijk}F = \frac{\max}{T_h}\left|F^{(i,j,k)}(\kappa,\upsilon,\zeta)\right|$.

Proof. Since $de\kappa(B^{\kappa\upsilon}_{m,q})=1$, by Peano's Kernel theorem, one obtain

$$(R^{\kappa\upsilon}_{m,q}F)(\kappa,\upsilon,\zeta) = \int_0^{h-\upsilon-\zeta} K_{200}(\kappa,\upsilon,\zeta,t)F^{(2,0,0)}(t,\upsilon,\zeta)dt,$$

where the kernel

$$K_{200}(\kappa,\upsilon,\zeta,t) := R^{\kappa\upsilon}_{m,q}[(\kappa-t)_+] = (\kappa-t)_+ - \sum_{i=0}^{m} p_{m,i}(\kappa,\upsilon,t)\left(\frac{[i]_q}{[m]_q}(h-\upsilon-\zeta)-t\right)_+$$

retains the same sign $K_{200}(\kappa,\upsilon,\zeta,t)\le 0, t\in[0,h-\upsilon-\zeta]$.

It follows from the Mean Value Theorem that

$$(R^{\kappa\upsilon}_{m,q}F)(\kappa,\upsilon,\zeta) = F^{(2,0,0)}(\xi,\upsilon,\zeta)\int_0^{h-\upsilon-\zeta} K_{200}(\kappa,\upsilon,\zeta;t)dt;\xi\in[0,h-\upsilon-\zeta]$$

We obtain by doing a simple computation for $\xi\in[0,h-\upsilon-\zeta]$

$$(R^{\kappa\upsilon}_{m,q}F)(\kappa,\upsilon,\zeta) = -\left(\frac{\kappa(h-\upsilon-\zeta-\kappa)}{2[m]_q} - \frac{\kappa^2(h-\upsilon-\zeta-\kappa)(1-q)}{2(h-\upsilon-\zeta-\kappa+q\kappa)}\left(1-\frac{1}{[m]_q}\right)\right)F^{(2,0,0)}(\xi,\upsilon,\zeta);$$

If $0<q\leqslant 1$, then

$$\frac{\kappa(h-\upsilon-\zeta-\kappa)}{2[m]_q} \ge \frac{\kappa^2(h-\upsilon-\zeta-\kappa)(1-q)}{2(h-\upsilon-\zeta-\kappa+q\kappa)}\left(1-\frac{1}{[m]_q}\right), \text{for all } (\kappa,\upsilon,\zeta)\in T_1$$

Now,

$$\left|(R_{m,q}^{\kappa\upsilon}F)(\kappa,\upsilon,\zeta)\right| \leqslant \frac{h^2}{8[m]_q} M_{200}F$$

Remark 2. Accordingly, the assessments of the remainder in the following formula are demonstrated.

$$F = B_{n,q}^{\upsilon\kappa}F + R_{n,q}^{\upsilon\kappa}F$$

i.e. for $F(\kappa,\cdot,\zeta) \in C[0,h-\kappa-\zeta]$

$$\left|(R_{n,q}^{\upsilon\kappa}F)(\kappa,\upsilon,\zeta)\right| \leq \left(1+\frac{h}{2}\right) W\left(F(\kappa,\cdot,\zeta);\frac{1}{\sqrt{[n]_q}}\right)$$

respectively, for $F(\kappa,\cdot,\zeta) \in C^2[0,h]$.

$$\left|(R_{n,2}^{\upsilon\kappa}F)(\kappa,\upsilon,\zeta)\right| \leq \frac{h^2}{8[n]_q} M_{020}F \text{ on } T_1.$$

Product Operators

$$\begin{aligned} P_{mn,q}^{1} &= B_{m,q}^{\kappa\upsilon} \quad B_{n,q}^{\upsilon\kappa} \\ Q_{nm,q}^{1} &= B_{n,q}^{\upsilon\kappa} \quad B_{m,q}^{\kappa\upsilon} \end{aligned}$$

Theorem 2.4. For real valued F,

$$P'_{mn,q}F = F \text{ and } Q'_{nm,q}F = F \text{ on } \tau_3 \cup \sigma_{123}.$$

Proof. Equations (1) and (2) gives

$$\begin{aligned} (P_{mn,q}^{1}F)(0,0,\zeta) &= F(0,0,\zeta) \\ (P_{mn,q}^{1}F)(h-\upsilon-\zeta,\upsilon,\zeta) &= F(h-\upsilon-\zeta,\upsilon,\zeta) \end{aligned}$$

respectively

$$\begin{aligned} (Q_{nm,q}^{1}F)(0,0,\zeta) &= F(0,0,\zeta) \\ (Q_{nm,q}^{1}F)(h-\upsilon-\zeta,\upsilon,\zeta) &= F(h-\upsilon-\zeta,\upsilon,\zeta), \forall \upsilon,\zeta \in [0,n] \end{aligned}$$

Theorem 2.5. If $F(\cdot,\cdot,\zeta) \in C([0,h]\times[0,h]), q>0$ then

$$\begin{aligned} \left|(R_{mn,q}^{P^1}F)(\kappa,\upsilon,\zeta)\right| \leq 1 &+ \frac{1}{\delta_1}\sqrt{\frac{\kappa(h-\kappa-\upsilon-\zeta)}{[m]_{\text{II}}} - \frac{\kappa^2(h-\kappa-\upsilon-\zeta)}{(h-\kappa-\upsilon-\zeta+q\kappa)}\cdot\left(1-\frac{1}{[m]_q}\right)} \\ &+ \frac{1}{\delta_2}\sqrt{\frac{\kappa(h-\kappa-\upsilon-\zeta)}{[n]_q} - \frac{\kappa^2(h-\kappa-\upsilon-\zeta)(1-q)}{(h-\kappa-\upsilon-\zeta+q\upsilon)}\cdot\left(1-\frac{1}{[n]_q}\right)} \\ W(F(\cdot,\cdot,\zeta),\delta_1,\delta_2) \end{aligned}$$

If $0 < q \le 1$, then

$$|(R^{P^1}_{mn,q}F)(\kappa,\upsilon,\zeta)| \le (1+h)W\left(F(\cdot,\cdot,\zeta),\frac{1}{\sqrt{[m]_q}},\frac{1}{\sqrt{[n]_q}}\right) \text{ on } T_h.$$

Proof.

$$|(R^{P^1}_{mn,q}F)(\kappa,\upsilon,\zeta)| \le \left[\frac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n}P_{m,i}(\kappa,\upsilon,\zeta)q_{n,j}\left(\frac{[i]_q}{[n]_q}(h-\upsilon-\zeta),\upsilon,\zeta\right)\left|\kappa-\frac{[i]_q}{[m]_q}(h-\upsilon-\zeta)\right|\right.$$

$$+\frac{1}{\delta_2}\sum_{i=0}^{m}\sum_{j=0}^{n}P_{m,j}(\kappa,\upsilon,\zeta)q_{n,j}\left(\frac{[i]_q}{[n]_q}(h-\upsilon-\zeta),\upsilon,\zeta\right)\left|\upsilon-\frac{[j]_q\left([m]_q-[n]_q\right)h+\left[[i]_q\upsilon\right]}{[m]_q[n]_q}\right|$$

$$\left.+\sum_{i=0}^{m}\sum_{j=0}^{n}P_{m,i}(\kappa,\upsilon,\zeta)q_{n,j}\left(\frac{[i]_q}{[m]_q}(h-\upsilon-\zeta),\upsilon,\zeta\right)\right]W(F(0,\cdot,\zeta),\delta_1,\delta_2)$$

After some transformation one obtains

$$\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}(\kappa,\upsilon,\zeta)q_{n,j}\left(\frac{[i]_q}{[m]_q}(n-\upsilon-\zeta),\upsilon,\zeta\right)\left|\kappa-\frac{[i]_q}{[m]_q}(n-\upsilon-\zeta)\right|$$

$$\leqslant\sqrt{\frac{\kappa(h-\upsilon-\zeta-\kappa)}{[m]_q}-\frac{\kappa^2(h-\upsilon-\zeta-\kappa)}{(h-\upsilon-\zeta-\kappa+q\kappa)}\left(1-\frac{1}{[m]_q}\right)}$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}(\kappa,\upsilon,\zeta)q_{n,j}\left(\frac{[i]_q}{[m]_q}(h-\upsilon-\zeta),\upsilon,\zeta\right)\left|\upsilon-\frac{[j]_q}{[n]_q}\frac{\left([m]_q-[i]_q\right)h+[i]_q\upsilon}{[n]_q}\right|$$

$$\leqslant\sqrt{\frac{\upsilon(h-\upsilon-\zeta-\kappa)}{[n]_q}-\frac{\upsilon^2(h-\upsilon-\zeta-\kappa)(1-q)}{(h-\upsilon-\zeta-\kappa+q\upsilon)}\cdot\left(1-\frac{1}{[n]_q}\right)}$$

while $\displaystyle\sum_{i=0}^{m}\sum_{j=0}^{n}P_{m,i}(\kappa,\upsilon,\zeta)q_{n,j}\left(\frac{[i]_q}{[m]_q}(n-\upsilon-\zeta),\upsilon,\zeta\right)=1.$

It follows

$$|(R^{P^1}_{mn,q}F)(\kappa,\upsilon,\zeta)| \le \left[\frac{1}{\delta_1}\cdot\sqrt{\frac{\kappa(h-\upsilon-\zeta-\kappa)}{[m]_q}-\frac{\kappa^2(h-\upsilon-\zeta-\kappa)}{(h-\upsilon-\zeta-\kappa+q\kappa)}\left(1-\frac{1}{[m]_q}\right)}\right.$$

$$\left.+\frac{1}{\delta_2}\cdot\sqrt{\frac{\upsilon(h-\upsilon-\zeta-\kappa)}{[n]_q}-\frac{\upsilon^2(h-\upsilon-\zeta-\kappa)}{(h-\upsilon-\zeta-\kappa+q\upsilon)}\left(1-\frac{1}{[n]_q}\right)}+1\right]\cdot W\left(F(\cdot,\cdot,\zeta),\delta_1,\delta_2\right)$$

Taking into account that if $0 < q \leqslant 1$, then

$$|(R^{P^1}_{mn,q}F)(\kappa,\upsilon,\zeta)| \leqslant \left[\frac{1}{\delta_1}\sqrt{\frac{\kappa(h-\upsilon-\zeta-\kappa)}{[m]_q}}+\frac{1}{\delta_2}\sqrt{\frac{\upsilon(h-\upsilon-\zeta-\kappa)}{[n]_q}}+1\right]\cdot W(F(\cdot,\cdot,\zeta),\delta_1,\delta_2)$$

Since $\dfrac{\kappa(h-\upsilon-\zeta-\kappa)}{[m]_q}\leqslant\dfrac{h^2}{4[m]_q}$, $\dfrac{\upsilon(h-\upsilon-\zeta-\kappa)}{[n]_q}\leqslant\dfrac{h^2}{4[n]_q}$ on $[0,h-\upsilon-\zeta]$ and $[0,h-\kappa-\zeta]$.

We have

$$|(R^{P^1}_{mn,q}F)(\kappa,\upsilon,\zeta)| \le \left(\frac{h}{2\delta_1[m]_q}+\frac{h}{2\delta_2[n]_q}+1\right)\cdot W(F(\cdot,\cdot,\zeta),\delta_1,\delta_2)$$

$$|(R^{P}_{mn,q})F(\kappa,\upsilon,\zeta)| \le (1+h)W\left(F,\frac{1}{\sqrt{[m]_q}},\frac{1}{\sqrt{[n]_q}}\right)$$

for $\delta_1=\frac{1}{\sqrt{[m]_q}}$ and $\delta_2=\frac{1}{\sqrt{[n]_q}}$

Boolean Operators

$$\begin{aligned} S^1_{mn,q} &:= B^{\kappa\upsilon}_{m,q}\oplus B^{\upsilon\kappa}_{n,q}=B^{\kappa\upsilon}_{m,q}+B^{\upsilon\kappa}_{n,q}-B^{\kappa\upsilon}_{m,q}B^{\upsilon\kappa}_{n,q}\\ \text{and } T^1_{nm,q} &:= B^{\upsilon\kappa}_{n,q}\oplus B^{\kappa\upsilon}_{m,q}=B^{\upsilon\kappa}_{n,q}+B^{\kappa\upsilon}_{m,q}-B^{\upsilon\kappa}_{n,q}B^{\kappa\upsilon}_{m,q}\end{aligned}$$

Theorem 2.6. If F is a real valued function defined on T_h, then $S^1_{mn,q}F=F$ and $T^1_{nm,q}F=F$ on $\sigma_{013}\cup\sigma_{023}\cup\sigma_{123}$.

Theorem 2.7. If $F\in C(T_h)$, then

$$\begin{aligned} |(R^S_{mn,q}F)(\kappa,\upsilon,\zeta)| \leqslant &\left[1+\frac{1}{\delta_1}\sqrt{\frac{\kappa(h-\kappa-\upsilon-\zeta)}{[m]_q}-\frac{\kappa^2(h-\kappa-\upsilon-\zeta)}{(h-\kappa-\upsilon-\zeta+q\kappa)}\left(1-\frac{1}{[m]_q}\right)}\right]W(F(\cdot,\upsilon,\zeta,\delta_1))\\ &+\left[1+\frac{1}{\delta_2}\sqrt{\frac{\upsilon(h-\kappa-\upsilon-\zeta)}{[n]_q}-\frac{\upsilon^2(h-\kappa-\upsilon-\zeta)}{(h-\upsilon-\zeta-\kappa+q\upsilon)}\left(1-\frac{1}{[n]_q}\right)}\right]W(F(\kappa,\cdot,\zeta);\delta_2)\\ &+\left[1+\frac{1}{\delta_1}\sqrt{\frac{\kappa(h-\kappa-\upsilon-\zeta)}{[m]_q}-\frac{\kappa^2(h-\kappa-\upsilon-\zeta)}{(h-\kappa-\upsilon-\zeta+q\kappa)}\left(1-\frac{1}{[m]_q}\right)}\right.\\ &\left.+\frac{1}{\delta_2}\sqrt{\frac{\upsilon(h-\kappa-\upsilon-\zeta)}{[n]_q}}-\frac{\upsilon^2(h-\kappa-\upsilon-\zeta)}{(h-\upsilon-\zeta-\kappa+q\upsilon)}\left(1-\frac{1}{[n]_q}\right)\right]W(F(\cdot,\cdot,\zeta),\delta_1,\delta_2)\end{aligned}$$

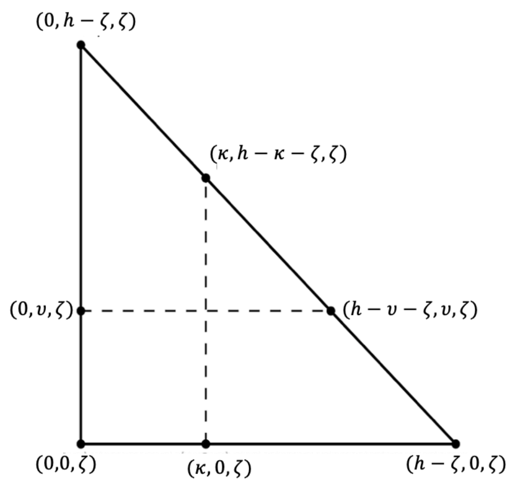

Figure 2: Triangle T_1

Moreover, if $\delta_1 = \frac{1}{\sqrt{[m]_q}}$ and $\delta_2 = \frac{1}{\sqrt{[n]_q}}$,

$$|(R^S_{mn,q}F)(\kappa,\upsilon,\zeta)| \leqslant 1+\frac{h}{2}W\left(F(\cdot,\upsilon,\zeta),\frac{1}{\sqrt{[m]_q}}\right)+\left(1+\frac{h}{2}\right)W\left(F(\kappa,\cdot,\zeta);\frac{1}{\sqrt{[n]_q}}\right)$$
$$+(1+h)W\left(F(\cdot,\cdot,\zeta);\frac{1}{\sqrt{[m]_q}},\frac{1}{\sqrt{[n]_q}}\right).$$

3. Lupaş Type Operators on Three Curved Edged Tetrahedrons

Tetrahedron with three curved edges and its triangle are shown in Figure 3 and Figure 4.

$$(B^{\kappa\upsilon}_{m,q}F)(\kappa,\upsilon,\zeta)=\sum_{i=0}^{m}P_{m,i}(\kappa,\upsilon,\zeta)F\left(\frac{[i]_q}{[m]_q}\sqrt{h^2-\upsilon^2-\zeta^2},\upsilon,\zeta\right)$$

and

$$(B^{\upsilon\kappa}_{n,q}F)(\kappa,\upsilon,\zeta)=\sum_{j=0}^{n}q_{n,j}(\kappa,\upsilon,\zeta)F\left(\kappa,\frac{[j]_q}{[n]_q}\sqrt{h^2-\kappa^2-\zeta^2},\zeta\right)$$

with $P_{m,i}(\kappa,\upsilon,\zeta)=\dfrac{\begin{bmatrix} m \\ i \end{bmatrix}_q q^{\frac{i(i-1)}{2}}\kappa^i\left(\sqrt{h^2-\upsilon^2-\zeta^2}-\kappa\right)^{m-i}}{\prod_{r=0}^{m-1}\left(\sqrt{h^2-\upsilon^2-\zeta^2}-\kappa+q^r\kappa\right)}$

respectively $q_{n,j}(\kappa,\upsilon,\zeta)=\dfrac{\begin{bmatrix} n \\ i \end{bmatrix}_q q^{\frac{j(j-1)}{2}}\upsilon^j\left(\sqrt{h^2-\kappa^2-\zeta^2}-\upsilon\right)^{n-j}}{\prod_{s=0}^{n-1}\left(\sqrt{h^2-\kappa^2-\zeta^2}-\upsilon+q^s\upsilon\right)}$.

Theorem 3.1. If $F: T_h \to \mathbb{R}$, then

i. $B^{\kappa\upsilon}_{m,q}F = F$ on $s_{023} \cup s_{123}$

   $B^{\upsilon\kappa}_{n,i}F = F$ on $s_{013} \cup s_{123}$

ii. $de\kappa(B^{\kappa\upsilon}_{m,q}) = de\kappa(B^{\upsilon\kappa}_{n,q}) = 1;$

iii. pres $(B^{\kappa\upsilon}_{m,q}) = \{\kappa^i\upsilon^j\zeta^k \mid i=0,1; j,k\in\mathbb{N}\}$

   pres $(B^{\upsilon\kappa}_{n,q}) = \{\kappa^i\upsilon^j\zeta^k \mid j=0,1; i,k\in\mathbb{N}\}$

iv. $(B^{\kappa\upsilon}_{m,q}e_{2jk})(\kappa,\upsilon,\zeta)=\left[\kappa^2+\frac{\kappa\left(\sqrt{h^2-\upsilon^2-\zeta^2}-\kappa\right)}{[m]_q}-\frac{\kappa^2\left(\sqrt{h^2-\upsilon^2-\zeta^2}-\kappa\right)(1-q)}{\sqrt{h^2-\upsilon^2-\zeta^2}-\kappa+q\kappa}\right]\left(1-\frac{1}{[m]_q}\right)\bigg]\cdot\upsilon^i\zeta^k$

$$(B^{\upsilon\kappa}_{n,q}e_{i2k})(\kappa,\upsilon,\zeta)=\left[\upsilon^2+\frac{\upsilon\sqrt{h^2-\kappa^2-\zeta^2}-\upsilon}{[n]_q}-\frac{\upsilon^2\left(\sqrt{h^2-\kappa^2-\zeta^2}-\upsilon\right)(1-q)}{\sqrt{h^2-\kappa^2-\zeta^2}-\upsilon+2\upsilon}\left(1-\frac{1}{[n]_q}\right)\right]\kappa^i\zeta^k$$

Theorem 3.2. If $F(\cdot,\upsilon,\zeta)\in C\left[0,\sqrt{h^2-\upsilon^2-\zeta^2}\right]$, then

$$|(R_{m,q}^{\kappa\upsilon}F)(\kappa,\upsilon,\zeta)|\le\left(1+\frac{h}{2\delta\sqrt{[m]_q}}\right)W(F(\cdot,\upsilon,\zeta);\delta),\upsilon+\zeta\le h$$

respectively for $\delta=\frac{1}{\sqrt{[m]_q}}$

$$|(R_{m,q}^{\kappa\upsilon}F)(\kappa,\upsilon,\zeta)|\leqslant\left(1+\frac{h}{2}\right)W\left(F(\cdot,\upsilon,\zeta);\frac{1}{\sqrt{[m]_q}}\right)$$

Theorem 3.3. If $F(\cdot,\upsilon,\zeta)\in C^2[0,h]$, then

$$(R_{m,q}^{\kappa\upsilon}F)(\kappa,\upsilon,\zeta)=-\left[\frac{\kappa\left(\sqrt{h^2-\upsilon^2-\zeta^2}-\kappa\right)}{2[m]_q}-\frac{\kappa^2\left(\sqrt{h^2-\upsilon^2-\zeta^2}-\kappa\right)}{2\left(\sqrt{h^2-\upsilon^2-\zeta^2-\kappa+q\kappa}\right)}\left(1-\frac{1}{[m]_q}\right)\right]F^{(2,0,0)}(\xi,\upsilon,\zeta)$$

for $0\le\xi\le\sqrt{h^2-\upsilon^2-\zeta^2},\upsilon,\zeta\in[0,h]$ and if $0<q\leqslant 1$, we have

$$|(R_{m,q}^{\kappa\upsilon}F)(\kappa,\upsilon,\zeta)|\le\frac{h^2}{8[m]_q}\cdot M_{200}F$$

Remark 4. Analogous results take place for the remainder in the approximation formula

$$F=B_{n,q}^{\upsilon\kappa}F+R_{n,q}^{\upsilon\kappa}F$$

Product Operators: Let $P_{mn}=B_{m,q}^{\kappa\upsilon}B_{n,q}^{\upsilon\kappa}$ and $Q_{nm}=B_{n,q}^{\upsilon\kappa}B_{m,q}^{\kappa\upsilon}$ be the products of the operators $B_{m,q}^{\kappa\upsilon}$ and $B_{n,i}^{\upsilon\kappa}$ i.e.

$$(P_{mn,q}F)(\kappa,\upsilon,\zeta)=\sum_{i=0}^{m}\sum_{j=0}^{n}P_{m,i}(\kappa,\upsilon,\zeta)q_{n,j}\left(\frac{[i]_q}{[m]_q}\sqrt{h^2-\upsilon^2-\zeta^2},\upsilon,\zeta\right)$$
$$F\left(\frac{[i]_q}{[m]_q}\sqrt{h^2-\upsilon^2-\zeta^2},\frac{[j]_q}{[m]_q[n]_q}\cdot\sqrt{\left([m]_q^2-[i]_q^2\right)(h^2-\zeta^2)+[i]_q\upsilon^2},\zeta\right)$$

Similarly,

$$(Q_{nm,q}F)(\kappa,\upsilon,\zeta)=\sum_{i=0}^{m}\sum_{j=0}^{n}P_{m,i}\left(\kappa,\frac{[i]_q}{[n]_q}\sqrt{h^2-\kappa^2-\zeta^2},\zeta\right)q_{n,j}(\kappa,\upsilon,\zeta)$$
$$\times F\left(\frac{[i]_q}{[m]_q[n]_q}\sqrt{([n]_q^2-j^2)(h^2-\zeta^2)+[j]_q^2\kappa^2},\frac{[j]_q}{[n]_q}\sqrt{h^2-\kappa^2-\zeta^2},\zeta\right)$$

Theorem 3.4. If $\mathrm{F}:T_h\to\mathbb{R}$, then $P_{mn,2}F=F$ and $Q_{nm,2}F=F$ on $\tau_3\cup S_{123}$.

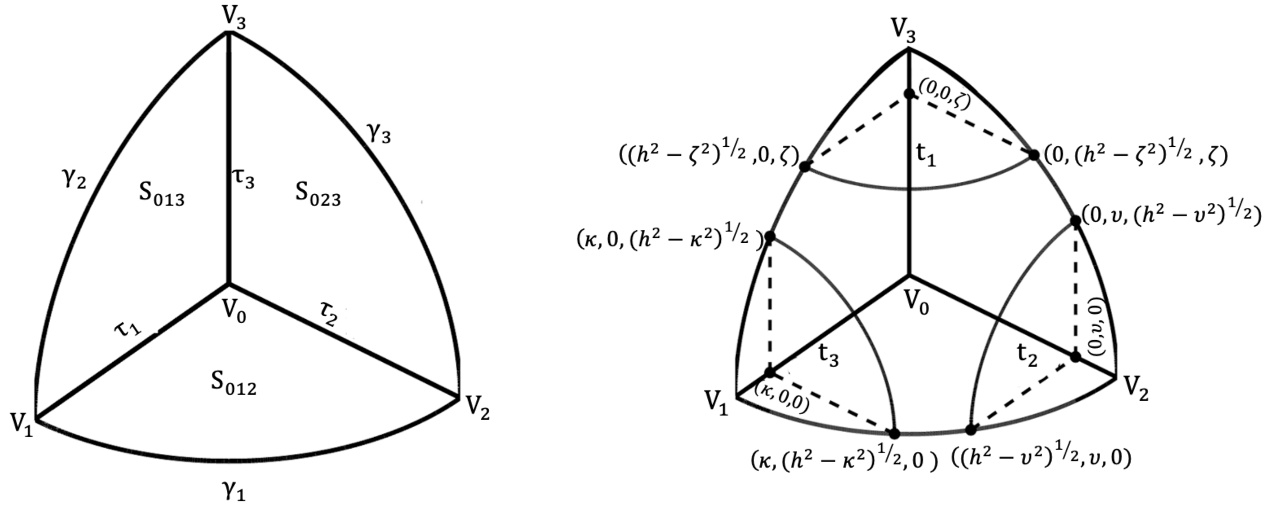

Figure 3: Tetrahedron with three curved edges [4]

Theorem 3.5. If $F(\cdot,\cdot,\zeta)\in C([0,h]\times[0,h])$, then

$$\left|(R^{p}_{mn,q}F)(\kappa,\upsilon,\zeta)\right|\le(1+h)W\left(F(0,0,\zeta);\frac{1}{\sqrt{[m]_q}},\frac{1}{\sqrt{[n]_2}}\right)$$

and

$$\left|(R^{Q}_{nm}F)(\kappa,\upsilon,\zeta)\right|\le(1+h)W\left(F(\cdot,0,\zeta);\frac{1}{\sqrt{[m]_q}},\frac{1}{\sqrt{[n]_q}}\right)$$

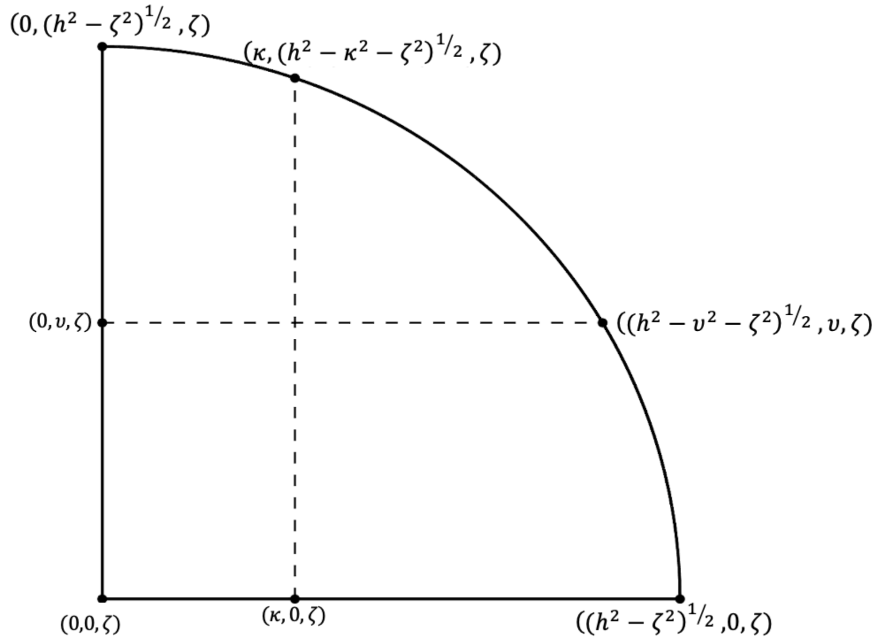

Figure 4: Triangle t_1 [4]

Bookan Sum Operators: If $S_{mn} = B^{\kappa\upsilon}_{m,q} \oplus B^{\upsilon\kappa}_{n,q}$ and $T_{nm,q} = B^{\upsilon\kappa}_{n,q} \oplus B^{\kappa\upsilon}_{m,q}$ are the Boolean sums of the operators $B^{\kappa\upsilon}_{m,q}$ and $B^{\upsilon\kappa}_{n,q}$, then we have

Theorem 3.6. If $F : T_h \to \mathbb{R}$, then $S_{mn}F = F$ and $T_{nm}F = F$ on $S_{a13}US_{023}US_{123}$.

Theorem 3.7. If $F \in C(T_h)$, then

$$|(F - S_{mn,q}F)(\kappa,\upsilon,\zeta)| \le \left(1+\frac{h}{2}\right)W\left(F(\cdot,\upsilon,\zeta);\frac{1}{\sqrt{[m]_q}}\right) + \left(1+\frac{h}{2}\right)W\left(F(\kappa,;\zeta);\frac{1}{\sqrt{[n]_q}}\right) + (1+h)W\left(F(\cdot,\cdot,\zeta);\frac{1}{\sqrt{[m]_q}}\cdot\frac{1}{\sqrt{[n]_q}}\right).$$

Similar procedure will yield the inequality in computing the error for $(F - T_{nm,q}F)$.

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