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C^*-algebra-valued measure of non-compactness with applications

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Abstract

In this article a novel concept of C^*-algebra-valued measure of non-compactness is defined. Using this concept, the well known fixed point theorems of Darbo, Sadovskii and Krasnoslskii are generalized. We present non-trivial examples and applications to validate the real utilization of our results.

Key words and phrases: Fixed point, Measure of non-compactness, Darbo fixed point theorem, C*-algebra valued measure of non-compactness.
Mathematics Subject Classification: 47H10, 54H25, 46L07, 46L05

1. Introduction

Topological fixed point theory is more important in the sense of application point of view. In metric fixed point theory the contractive conditions shrinks the operator's class, because a small class would be able to satisfy the contractive conditions. Schauder fixed point theorem [1] had been used for existence theory of linear and nonlinear operators with compact domains. Many questions were rosed on the compactness in this theorem, since there is a huge class of operators under non-compact domains. This problem was first addressed by Darbo [2], using the notions of measure of non-compactness defined by Kuratowski [3] and Istratescu [4]. The following are the theorems of Schauder and Darbo.

Theorem 1.1: [1] *Let E be a Banach space and let Ω be a compact and convex subset of E. Then, any continuous operator $F:\Omega\to\Omega$ has at least one fixed point.*

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Theorem 1.2: [2] *Let E be a Banach space and let Ω be a bounded, closed and convex subset of E. Then, any continuous operator k-set contraction has at least one fixed point.*

In the next section, we will define the k-set contraction. Many generalizations of Drabo's fixed point theorem by weakening the k-set contraction can be seen in the literature [5–9]. In 2016 Jleli et al., in [10] defined the notions of cone measure of non-compactness by replacing the set of reals with a real Banach space having normal cone. They proved fixed point theorem as a generalization of Darbo's theorem and proved existence results for functional integral equations. This concept is further generalized in this direction by Deng et al., in 2018. They have also generalized the results of Darbo and Sadowski [11]. The abstract E-metric space of related fixed points results were studied in [12–14] Recently Ma et al., [15] defined C^*-algebra valued metric space by replacing the set of real numbers $\mathbb{R}$ by a unital C^*-algebra. The next discussion about C^*-algebra has been taken from Ma et al. Inspired from them we introduced the notion of C^*-algebra valued measure of non-compactness.

First we recall the following definitions concerning C^*-algebra.

Definition 1.3: *A Banach algebra A is an algebra over $\mathbb{C}$ that has a norm $\|.\|$ relative to which A is Banach apace and $\|ab\| \le \|a\|\|b\|$ for all $a,b \in A$.*

Definition 1.4: *An involutive algebra A is a complex algebra given together with an involution $\begin{cases} A \to A \\ a \mapsto a^* \end{cases}$ such that:*

$$\begin{aligned}(a+b)^* &= a^* + b^*,\\ (\lambda a)^* &= \bar{\lambda} a^*,\\ (ab)^* &= b^* a^*,\\ (a^*)^* &= a.\end{aligned}$$

Definition 1.5: *A C^*-algebra is a Banach involutive algebra A over $\mathbb{C}$ satisfying the identity $\|a^*a\| = \|a\|^2$ for all $a \in A$.*

Definition 1.6: *Let A be a unital C^*-algebra. Then*

1. $a \in A$ is self-adjoint if $a^* = a$,
2. $a = a^*$ is positive if $\sigma(a) \subset [0,\infty)$, where $\sigma(a)$ denotes the spectrum of a.

In this paper, we define the notion of C^*-algebra valued measure of non-compactness and generalize the Darbo, Sadovskii and Krasnoselskii fixed point theorems. We finally present some applications to validate our results.

2. Main Results

Let $\mathcal{A}$ be a unital C^*-algebra. A self-adjoint element a in a C^*-algebra $\mathcal{A}$ is positive if $\sigma(a) \subset [0,\infty)$. Denote by $\mathcal{A}_+$ the set of all positive elements a of $\mathcal{A}$. This allow us to define partial ordering $\preceq$ on the self-adjoint elements of $\mathcal{A}$. For a,b self-adjoint, we say $a \preceq b$ if and only if $b - a \in \mathcal{A}_+$, and for $a \prec b$ means $a \preceq b$ with $a \neq b$, clearly $\mathcal{A}_+ = \{a \in \mathcal{A} : a \succeq 0\}$.

Now, let us recall the definition [[16], Definition 3.1.3], before defining C^*-algebra valued measure of non-compactness.

Let E be a Banach space. The symbol $\overline{X}$, $ConvX$ denotes the closure and closed convex hull of a subset X of E, respectively. Moreover, $\mathfrak{M}_E$ indicate the family of all non-empty and bounded subsets of E and $\mathfrak{N}_E$ indicate the family of all nonempty and relatively compact subsets.

Definition 2.1: [16] *The function $\mu : \mathfrak{M}_E \to [0,+\infty]$ is said to be a measure of non-compactness if it satisfy the following conditions:*

1. The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is non-empty and $ker\mu \subseteq \mathfrak{N}_E$,
2. $\mu(\overline{X}) = \mu(X)$,

3. $X \subset Y$ ifandonlyif $\mu(X) \leq \mu(Y)$,
4. $\mu(ConvX) = \mu(X)$,
5. $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.
6. If $X_n \in E$, $X_n = \overline{X}_n$ and $X_{n+1} \subset X_n$, for $n = 1,2,....$ satisfying $\lim_{n\to\infty} \mu(X_n) = 0$, implies $\bigcap_{n=0}^{\infty} X_n := X_\infty$ is nonempty.

Let E be a Banach space, and let $\sum$ denote the set of all bounded subsets of E. Then, we define the C^*-algebra valued measure of non-compactness as follows:

Definition 2.2: *Let E be a Banach space. A mapping $\kappa : \sum \to \mathcal{A}_+$ is called C^*-algebra valued measure of non-compactness, if the following conditions hold:*

($MC1$) For $\Omega \in \sum$, $\kappa(\Omega) = 0$ implies Ω is pre-compact subset of E;

($MC2$) For $\Omega_1, \Omega_2 \in \sum$ with $\Omega_1 \subseteq \Omega_2$ implies $\kappa(\Omega_1) \preceq \kappa(\Omega_2)$;

($MC3$) For $\Omega \in \sum$, $\kappa(\overline{\Omega}) = \kappa(\Omega)$;

($MC4$) For $\Omega \in \sum$, $\kappa(co(\Omega)) = \kappa(\Omega)$, where co denotes the convex hull;

($MC5$) For a decreasing sequence of sets $\{\Omega_m\} \subset \sum$ satisfying $\lim_{m\to\infty} \kappa(\Omega_m) = 0$, implies $\bigcap_{m=0}^{\infty} \Omega_m := \Omega_\infty$ is nonempty;

($MC6$) $\kappa(\theta\Omega_1 + (1-\theta)\Omega_2) \leq \theta\kappa(\Omega_1) + (1-\theta)\kappa(\Omega_2)$ for $\theta \in [0,1]$.

Remark 2.3: *Let E be a Banach space, and let $\sum$ denote the set of all bounded subsets of E. If the C^*-algebra $\mathcal{A} = \mathbb{C}$, then the C^*-algebra-valued measure of non-compactness is an ordinary measure of non-compactness μ (defined in Definition 2.1), as $\mathcal{A}_+$ of $\mathbb{C}$ is* $[0,+\infty)$.

Now, let us explain our definition with examples.

Example 2.4: *Let $E = \mathbb{R}$ and consider the C^*-algebra $\mathcal{A} = \mathbb{M}_{n\times n}(\mathbb{R})$ with* $||A|| = max\{||Ax|| : x \in \mathbb{R}, ||x|| = 1\}$, *where A is the matrix in C^*-algebra $\mathcal{A}$ with the partial ordering $A \preceq B$ iff $B - A \in \mathcal{A}^+$. Let α denote the Kuratowski* [3] *measure of non-compactness of non-empty and bounded subset Q of a complete metric space (X,d), given by*

$$\alpha(Q) = \inf\left\{\lambda > 0 : Q \subset \bigcup_{i=0}^{n} S_i, \delta(S_i) \leq \lambda\right\},$$

where δ denotes the diameter of the set.

Let $\sum$ denote the set of all bounded subsets of $\mathbb{R}$. Then, for $\Omega \in \sum$, we define

$$\kappa : \sum \to \mathcal{A}^+ \text{ by}$$
$$\kappa(\Omega) = diag(k_1 \cdot \alpha(\Omega), k_2 \cdot \alpha(\Omega), \ldots, k_n \cdot \alpha(\Omega)),$$

where $k_i > 0$, $i = 1,2,3,...,n$ are constants, and " diag" denotes the diagonal matrix of the given n-tuple(i.e., n-tuple is diagonal of $n \times n$ matrix and all other elements are zero). Then, κ is a C^*-algebra valued (i.e., matrix valued) measure of non-compactness.

Example 2.5: *Let $E = L^\infty(\Omega)$ and $\mathcal{H} = L^2(\Omega)$, where Ω is a Lebesgue measurable set. The set of all bounded linear operators on $\mathcal{H}$ will be denoted by $B(\mathcal{H})$. Then, $B(\mathcal{H})$ is a C^*-algebra with usual norm defined on operators. Let $\sum$ be the set of all bounded subsets of $L^\infty(\Omega)$. Then, define*

$$\kappa : \Sigma \to B(\mathcal{H}) \text{ by}$$
$$\kappa(\Omega) = \mathfrak{M}(\Omega).f \quad \text{for some } 0 \neq f \in \mathcal{H},$$

where $\mathfrak{M}$ is the measure of non-compactness defined on L^{∞} in [18]. Then, κ is a C^*-algebra valued (i.e., function valued) measure of non-compactness.

Now, we define a crucial definition to prove our first main result for finding fixed points of mappings using the notion of C^*-algebra valued measure of non-compactness.

Definition 2.6: *Let E be a Banach space and Ω be a closed bounded and convex subset of E. A mapping $F:\Omega\to\Omega$ is called A-set contraction if there exists $A\in\mathcal{A}$ with $\|A\|<1$ such that*

$$\kappa(F(C))\preceq A\kappa(C)A^* \tag{A}$$

for $C\in\Omega$.

Next, we prove our main result to generalize the fixed point results of Schauder and Darbo [1, 2].

Theorem 2.7 *Every continuous A-set contraction F on a Banach space E has a fixed point.*

Proof. Define a sequence $\{C_n\}\subset\Omega$ such that $C_0=C,\ C_1=co(F(C_0)),C_2=co(F(C_1)),\ldots,C_{n+1}=co(F(C_n))$.

Clearly $\{C_n\}$ is a decreasing sequence. Consider

$$\kappa(C_{n+1})=\kappa(co(F(C_n)))=\kappa(F(C_n)) \text{ for all } n\geq 0,$$

therefore using (A) with properties of κ we have

$$\begin{aligned}\kappa(C_{n+1})&=\kappa(F(C_n))\\&\preceq A\kappa(C_n)A^*\\&\preceq A(A\kappa(C_{n-1})A^*)A^*=A^2\kappa(C_{n-1})A^{*2}\\&\ \vdots\\&\preceq A^n\kappa(C_0)A^{*n}\\&\preceq\left\|\kappa(C_0)^{\frac{1}{2}}\right\|^2\left\|A^n\right\|^2 I\\&\preceq\left\|\kappa(C_0)^{\frac{1}{2}}\right\|^2\left\|A\right\|^{2n} I\to 0 \text{ as } n\to\infty.\end{aligned}$$

Hence, from $(MC5)$, $\bigcap_{n=0}^{\infty}C_n:=C_\infty$ is nonempty. Now, as

$$\kappa(C_\infty)=\kappa\left(\bigcap_{n=0}^{\infty}C_n\right)\preceq\kappa(C_n)=0,$$

therefore C_∞ is nonempty and compact. Since F is continuous, therefore by consequence of Schauder's fixed point theorem, F has a fixed point.

Definition 2.8: *Let E be a Banach space and Ω be a closed bounded and convex subset of E. A mapping $F:\Omega\to\Omega$ is called C^*-condensing if*

$$\kappa(F(C))\prec\kappa(C) \tag{B}$$

for any $C\in\Omega$.

The next theorem is a variant of Sadowskii's fixed point theorem using C^*-algebra valued measure of non-compactness.

Theorem 2.9 *Let E be a Banach space and Ω be a closed bounded and convex subset of E. If there exists some $e_0\in\Omega$ such that*

$$\kappa(Y \cup \{e_0\}) = \kappa(Y) \text{ for } Y \in \Sigma.$$

Let $F : \Omega \to \Omega$ be a continuous C^*condensing operator. Then, F has a fixed point.

Proof. Define

$$\Gamma = \{W \in \Sigma(\Omega) : e_0 \in W \text{ and } F(W) \subseteq W\}.$$

Clearly, $\Omega \in \Gamma$ and Γ is nonempty. Now, set $\mathcal{Y} = \bigcap_{W \in \Sigma(\Omega)} W$. We can check easily that $e_0 \in \mathcal{Y}$ and $\mathcal{Y}$ is closed and convex. Also $F(\mathcal{Y}) \subseteq \mathcal{Y}$, if we take $Q = con(F(\mathcal{Y}) \cup \{e_0\})$. Now, claim that $\mathcal{Y} = Q$. For this, as $e_0 \in \mathcal{Y}$ and $F(\mathcal{Y}) \subseteq \mathcal{Y}$, therefore $Q \subseteq \mathcal{Y}$. For reverse inclusion $Q \subseteq \mathcal{Y}$ implise $F(\mathcal{Y}) \subseteq F(\mathcal{Y}) \subset Q$ and $e_0 \in Q$ implies that $Q \in \Gamma$. Hence $\mathcal{Y} \subseteq Q$, which proves $\mathcal{Y} = Q$. Using hypothesis of theorem, consider

$$\begin{aligned}\kappa(\mathcal{Y}) &= \kappa(F(\mathcal{Y}) \cup \{e_0\}) \\ &= \kappa(F(\mathcal{Y})).\end{aligned}$$

Now to prove $\kappa(\mathcal{Y}) = 0$, suppose on contrary that $\kappa(\mathcal{Y}) \neq 0$ then by the definition of C^*-condensing $\kappa(\mathcal{Y}) = \kappa(F(\mathcal{Y})) \prec \kappa(\mathcal{Y})$ a contradiction, so $\kappa(\mathcal{Y}) = 0$. Since $\mathcal{Y}$ is pre-compact and closed, so is compact. Hence the application of Schauder's fixed point theorem the operator $F \downarrow_{\mathcal{Y}}$ has a fixed point. This proves the theorem.

The importance of Krasnoselskii fixed point theorem is obvious for the existence of solutions of perturbed and neutral differential operators. The next theorem is the version of Krasnoselskii fixed point theorem in the setting of C^*-algebra valued measure of non-compactness.

Theorem 2.10: *Let E be a Banach space and Ω be a closed bounded and convex subset of E. Let $S, F : \Omega \to \Omega$ be continuous operators such that*

(a) S is an A-set contraction;
(b) F is compact;
(c) $(S + F)\Omega \subset \Omega$.

Then the solution of operator equation $v = Sv + Fv$ exists in Ω.

Proof. Set $G = S + F$, we show that G is an A-set contraction. For this let $C \subset \Omega$ and consider

$$\begin{aligned}\kappa(G(C)) &= \kappa((S + F)(C)) \\ &\preceq \kappa(S(C)) + \kappa(F(C)) \\ &\preceq \kappa(S(C)) + 0 \quad \text{using } (b) \\ &\preceq A\kappa(C)A^* \quad \text{using } (a)\end{aligned}$$

which implies

$$\kappa((S + F)(C)) \preceq A\kappa(C)A^*,$$

Since G is continuous A-set contraction, therefor it has a fixed point say $z = G(z) = Sz + Fz$. This proves the theorem.

Definition 2.11: *A bounded sequence $\{a_n\}$ in a C^*-algebra $\mathcal{A}$ is called central sequence such that*

$$\| a_n x - x a_n \| \to 0 \text{ as } n \to \infty.$$

Theorem 2.12: *Let E be a Banach space and Ω be a closed bounded, convex subset of E. Let $F : \Omega \to \Omega$ be a continuous operator then there exist $y_0 \in \Omega$ such that for all $\lambda \in (0,1)$ and $Y \in \Sigma$,*

1. $\kappa(\lambda FY + (1-\lambda)\{y_0\}) \preceq \lambda\kappa(FY)$
2. $(I-F)\Omega$ is closed where $I:\Omega \to \Omega$ is identity map,
3. $\kappa(FY) \preceq \kappa(Y)$,

Then, F has a fixed point.

Proof. Let $\{\lambda_n\}$ be a sequence in (0,1) such that $lim_{n\to\infty}\lambda_n = 1$ and F_n be the sequence of mappings defined by $F_n : \Omega \to \Omega$ by

$$F_n y = \lambda_n Fy + (1-\lambda_n)y_0 \quad \text{for all } \; y \in \Omega, n = 0,1,2,..$$

Now, since Ω is convex therefore F_n is well-defined. Assume for all $y \in \Sigma$ and $n = 0,1,2...$

$$\begin{aligned}\kappa(F_n Y) &= \kappa(\lambda_n FY + (1-\lambda_n)\{y_0\}) \\ &\preceq \lambda_n \kappa(FY) \\ &\preceq \lambda_n \kappa(Y).\end{aligned}$$

Now, define another sequence $G_n : E \to E$ by

$$G_n v = \lambda_n v, \quad v \in E, \quad n = 0,1,2,..$$

clearly $G_n \in L(\mathcal{A})$ for $n = 0,1,2,...$ and is A-set contraction, Hence by Theorem 2.7 the mapping F_n has a fixed point $y_n \in \Omega$ for $n = 0,1,2,...$ i.e

$$\begin{aligned}F_n y_n &= \lambda_n F y_n + (1-\lambda_n)y_0 \\ &= y_n.\end{aligned}$$

Therefore,

$$\begin{aligned}(I-F)y_n &= F_n y_n - F y_n \\ &= (\lambda_n - 1)F y_n + (1-\lambda_n)y_0.\end{aligned}$$

Consequently, as $\{Fy_n\}$ is a bounded sequence, we get that $\{(I-F)y_n\}$ is a central sequence . Now, given that $(I-F)\Omega$ is closed therefore $0 \in (I-F)\Omega$. Hence there exist $y \in \Omega$ such that $(I-F)y = 0$ and then the theorem has been proved.

3. Application

In this section, as an application of C^*-algebra valued measure of non-compactness, we will prove the existence of solutions of a certain type of integral equation. Define:

(i) $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$, and
(ii) $\alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$. Where η and g are continuous. Let the following integral equation:

$$\psi(s) = g(s) + \int_0^{\alpha(s)} \eta(s,t,\psi(\beta(s)))dt$$

for s belongs to positive real numbers. Now, define $H := B_C(\mathbb{R}^+, \mathbb{R})$, the normed linear space of all bounded real valued functions with supremum norm defined on $\mathbb{R}^+$ with supremum norm. The space $\left(H, \|\cdot\|_{\mathbb{R}}\right)$ is a regular ordered Banach space, for details see [19].

We define a C^*-valued measure of non-compactness as follows. Let $L(H)$ be the space of all bounded linear operators on H. Clearly $L(H)$ is a unital C^*-algebra with identity I and usual operator norm [15]. Let $N \in H$ be a bounded convex set, define

$$N(s) = \{\psi(s) : s > 0, \psi \in N\}$$

and

$$\kappa(N) = \lim_{s\to\infty}\sup \delta(N(s)),$$

where $\delta(N(s)) = \sup\{|\psi(s) - \omega(s)| : \psi, \omega \in N\}$, define

$$\kappa_E(N) = \kappa(N)I$$

for $I \in L(H)$. Then clearly κ_E is a C^*-algebra valued measure of non-compactness. Let us consider the following;

($A1$) g, β are continuous.

($A2$) There exists functions $x, y : \mathbb{R}^+ \to \mathbb{R}^+$ such that η is continuous with

$$|\eta(s,t,\psi)| \leq x(s)y(t), \text{for all } t,s \in \mathbb{R}^+,$$

and $\lim_{s\to\infty} x(s)\int_0^{\alpha(s)} y(t)dt < 1$. Now we will consider the following theorem:

Theorem 3.1: *Assume that* ($A1$) *and* ($A2$) *are true. Then the following integral equation*

$$\psi(s) = g(s) + \int_0^{\alpha(s)} \eta(s,t,\psi(\beta(s)))dt$$

has at least one solution v^* in H.

Proof. We define

$$\Gamma\psi = \int_0^{\alpha(s)} \eta(s,t,\psi(\beta(s)))dt + g(s)$$

then using ($A2$), Γ is bounded. Since η is continuous so Γ is also continuous. Now, to prove that Γ is A –set contraction, let for $\psi, \omega \in N$, a bounded and convex set in H, consider

$$\begin{aligned}|\Gamma\psi(s) - \Gamma\omega(s)| &= \left|\int_0^{\alpha(s)} \eta(s,t,\psi(\beta(s)))dt - \int_0^{\alpha(s)} \eta(s,t,\omega(\beta(s)))dt\right| \\ &\leq x(s)\int_0^{\alpha(s)} y(t)|\psi(t) - \omega(t)|dt \\ &\leq \ell\delta(N(s)),\end{aligned}$$

where $\ell = \sup\left\{\lim_{v\to\infty} x(s)\int_0^{\alpha(s)} y(t)dt\right\} < 1$, using ($A2$), now we get

$$\delta(\Gamma(N(s))) \leq \ell\delta(N(s),$$

for all $s \in \mathbb{R}^+$. Let $\lim\sup$ over $s \to \infty$, we have

$$\kappa(\Gamma(N))I \preceq \ell(\kappa(N))I,$$

or

$$\kappa(\Gamma(N))I \preceq \left(\ell^{\frac{1}{2}}I\right)^* \kappa(N)I\left(\ell^{\frac{1}{2}}I\right)$$

and finally

$$\kappa_E(\Gamma(N)) \preceq \left(\ell^{\frac{1}{2}}I\right)^* \kappa_E(N)\left(\ell^{\frac{1}{2}}I\right).$$

Clearly $\left\|\ell^{\frac{1}{2}}I\right\| < 1$. Hence, properties of Theorem 2.7, are checked to obtain the fixed point of Γ.

Theorem 3.2: *Suppose $L(\mathcal{H})$ be the space of all bounded linear operators on Hilbert space $\mathcal{H}$. Let $A_1, A_2, ..., A_n \in L(\mathcal{H})$ with $\sum_{i=1}^{\infty} ||A_n||^2 \leq 1$. For $\Omega \in L(\mathcal{H})$ and $Q \in L(\mathcal{H})_+$. Then*

$$\Omega - \sum_{n=1}^{\infty} A_n^* \Omega A_n = Q$$

has at least one solution in $L(\mathcal{H})$.

Proof. Let $l = \sum_{i=1}^{\infty} ||A_n||^2$ for any $l > 0$. Now let I be the identity operator with usual norm in $L(\mathcal{H})$. For a bounded convex subset N in $L(\mathcal{H})$ define:

$$N(s) = \{\psi(s) : s > 0, \psi \in N\} \text{ for all } s \in \mathbb{R}^+,$$

and

$$\kappa(N) = \limsup_{s \to \infty} \delta(N(s)),$$

where $\delta(N(s)) = \sup\left\{|\psi(s) - \omega(s)| : \psi, \omega \in N\right\}$, define

$$\kappa_E(N) = \kappa(N)I$$

for $I \in L(\mathcal{H})$. Then clearly κ_E is a C^*-valued measure of non-compactness.

Consider a map $F : L(\mathcal{H}) \to L(\mathcal{H})$ defined by

$$F(\Omega) = \sum_{i=1}^{\infty} A_n^* \Omega A_n + Q$$

then

$$\begin{aligned} |F(\Omega) - F(\mu)| &= \left\| \sum_{i=1}^{\infty} A_n^* \Omega A_n - \sum_{i=1}^{\infty} A_n^* \mu A_n \right\| \\ &= \left\| \sum_{i=1}^{\infty} A_n^* (\Omega - \mu) A_n \right\| \\ &\leq \sum_{i=1}^{\infty} \|A_n\|^2 \|\Omega - \mu\| \\ &\leq l \|\Omega - \mu\| \\ &\leq l\delta(N(s)) \end{aligned}$$

where $l = \sum_{i=1}^{\infty} \|A_n\|^2 < 1$. Now using $(A2)$ we get,

$$\delta F(N(s)) \leq l\delta(N(s)), \text{for all } s \in \mathbb{R}^+.$$

Now taking $\limsup$ over $s \to \infty$, we get

$$\kappa(F(N))I \preceq l\kappa(N)I,$$

or

$$\kappa(F(N))I \preceq \left(l^{\frac{1}{2}}I\right)^{*} \kappa(N)I\left(l^{\frac{1}{2}}I\right)$$

Hence

$$\kappa_E(F(N)) \preceq \left(l^{\frac{1}{2}}I\right)^{*} \kappa_E(N)\left(l^{\frac{1}{2}}I\right).$$

Clearly $\left\|l^{\frac{1}{2}}I\right\| < 1$, and the conditions of Theorem 2.7, has been checked. Hence F has a fixed point.

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