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Liftings from Lorentzian para-Sasakian manifolds to its tangent bundle

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Abstract

The subject of the present study is to investigate liftings from Lorentzian para-Sasakian manifolds to its tangent bundle.

Key words and phrases: Complete lift; Tangent bundle; Weyl conformal curvature tensor; Mathematical operators; Quasi conformal curvature tensor; Partial differential equations.

Mathematics Subject Classification (2020): 53C15, 53C25, 53C40, 58A30

1. Introduction

Matsumoto [1] first proposed the idea of a Lorentzian para-Sasakian (briefly, LP-Sasakain) manifold in 1989. The same idea was then independently suggested by Mihai and Rosca [2], who produced a number of findings in this manifold. De et. al. [3–5], Khan [6], Matsumoto and Mihai [7], Sato [8] and Tarafdar and Bhattacharya [9] have also explored LP-Sasakian manifolds. The Weyl conformal curvature tensor $\mathcal{C}$ and the concircular curvature tensor $\tilde{\mathcal{C}}$ on a Riemannian manifold M of dimension n have been studied by Adati and Matsumoto [10], Chaki and Gupta [11] and Yano and Sawaki [12].

Let $\mathfrak{u}_1, \mathfrak{u}_2$ and $\mathfrak{u}_3$ be vector fields on M with a metric g. The Weyl conformal curvature tensor $\mathcal{C}$ is given by

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$$\begin{aligned}\mathcal{C}(\mathfrak{u}_1,\mathfrak{u}_2)\mathfrak{u}_3 = {} & R(\mathfrak{u}_1,\mathfrak{u}_2)\mathfrak{u}_3 \\ & -\frac{1}{n-2}\{g(\mathfrak{u}_2,\mathfrak{u}_3)Q\mathfrak{u}_1 - g(\mathfrak{u}_1,\mathfrak{u}_3)Q\mathfrak{u}_2 + S(\mathfrak{u}_2,\mathfrak{u}_3)\mathfrak{u}_1 - S(\mathfrak{u}_1,\mathfrak{u}_3)\mathfrak{u}_2\} \\ & +\frac{r}{(n-1)(n-2)}\{g(\mathfrak{u}_2,\mathfrak{u}_3)\mathfrak{u}_1 - g(\mathfrak{u}_1,\mathfrak{u}_3)\mathfrak{u}_2\},\end{aligned} \tag{1.1}$$

where $S(\mathfrak{u}_1,\mathfrak{u}_2) = g(Q\mathfrak{u}_1,\mathfrak{u}_2)$ and S,Q,R,r on M denote the Ricci tensor, the Ricci operator, the curvature tensor and the scalar curvature, respectively.

The quasi conformal curvature tensor $\tilde{\mathcal{C}}$ is given by

$$\begin{aligned}\tilde{\mathcal{C}}(\mathfrak{u}_1,\mathfrak{u}_2)\mathfrak{u}_3 = {} & \mathfrak{a}R(\mathfrak{u}_1,\mathfrak{u}_2)\mathfrak{u}_3 \\ & -\mathfrak{b}\{S(\mathfrak{u}_2,\mathfrak{u}_3)\mathfrak{u}_1 - S(\mathfrak{u}_1,\mathfrak{u}_3)\mathfrak{u}_2 + g(\mathfrak{u}_2,\mathfrak{u}_3)Q\mathfrak{u}_1 - g(\mathfrak{u}_1,\mathfrak{u}_3)Q\mathfrak{u}_2\} \\ & -\frac{r}{n}\left(\frac{\mathfrak{a}}{n-1} + 2\mathfrak{b}\right)\{g(\mathfrak{u}_2,\mathfrak{u}_3)\mathfrak{u}_1 - g(\mathfrak{u}_1,\mathfrak{u}_3)\mathfrak{u}_2\},\end{aligned} \tag{1.2}$$

where $\mathfrak{a},\mathfrak{b}$ are constants and $\mathfrak{a}\mathfrak{b} \neq 0$ and $S(\mathfrak{u}_1,\mathfrak{u}_2) = g(Q\mathfrak{u}_1,\mathfrak{u}_2)$.

On the other hand, one of the primary contributions of differential geometry of tangent bundles is to allows an effective differential geometry domain. Yano and ishihara [13] introduced notion of liftings of tensor fields and connections to tangent bundles and established some properties of curvature tensors. Dida and Hathout ([14–16]) determined Ricci soliton structures with lift torqued potential fields on tangent bundles of Riemannian manifolds. Numerous investigators [17–23] have studied several connections and geometric structures on the tangent bundle and providing their ideas.

The proposed paper's objective is to investigate the liftings from Lorentzian para-Sasakian manifolds to its tangent bundle. The followings are a compilation of the paper's main findings:

- Lifts of LP-Sasakian manifolds with the Weyl conformal curvature tensor
$$\mathcal{C} = 0 \tag{1.3}$$
are the subject of our investigation.
- A conformally flat LP-Sasakian manifold on TM is locally isometric to a unit sphere $S^n(1)$ has shown.
- Lifts of LP-Sasakian manifolds with the quasi conformal curvature tensor
$$\tilde{\mathcal{C}} = 0 \tag{1.4}$$
are the subject of our investigation.
- A quasi conformally flat LP-Sasakian manifold on TM is locally isometric to a unit sphere $S^n(1)$ has shown.

Notations: Throughout the article following notations are used: $\Im_r^s(M)$ and $\Im_r^s(TM)$ denote the set of all tensor fields of type (r,s), that is of contravariant degree r and covariant degree s, in M and TM, respectively.

2. Preliminaries

A differentiable manifold M (dim$=n$) is called an LP-Sasakian ([1], [2]) if it allows a (1, 1)-tensor field ϕ, a vector field ξ, a 1-form η and a Lorentzian metric g which suffice

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 = I + \eta \otimes \xi, \tag{2.2}$$

$$g(\phi\mathfrak{u}_1,\phi\mathfrak{u}_2)=g(\mathfrak{u}_1,\mathfrak{u}_2)+\eta(\mathfrak{u}_1)\eta(\mathfrak{u}_2) \tag{2.3}$$

$$g(\mathfrak{u}_1,\xi)=\eta(\mathfrak{u}_1),\quad \nabla_{\mathfrak{u}_1}\xi=\phi\mathfrak{u}_1, \tag{2.4}$$

$$(\nabla_{\mathfrak{u}_1}\phi)\mathfrak{u}_2=g(\mathfrak{u}_1,\mathfrak{u}_2)\xi+2\eta(\mathfrak{u}_1)\eta(\mathfrak{u}_2)\xi+\mathfrak{u}_1\eta(\mathfrak{u}_2), \tag{2.5}$$

where ∇ indicates the operator of covariant differentiation with respect to the Lorentzian metric g.

The relationships listed below hold in an LP-Sasakian manifold:

$$\phi\xi=0,\quad \eta(\phi\mathfrak{u}_1)=0, \tag{2.6}$$

$$rank\phi=n-1. \tag{2.7}$$

an LP-Sasakian manifold M is called η-Einstein if

$$S(\mathfrak{u}_1,\mathfrak{u}_2)=\mathfrak{a}g(\mathfrak{u}_1,\mathfrak{u}_2)+\mathfrak{b}\eta(\mathfrak{u}_1)\eta(\mathfrak{u}_2),\forall\mathfrak{u}_1,\mathfrak{u}_2\in\Im_0^1(M), \tag{2.8}$$

where S is Ricci tensor and $\mathfrak{a},\mathfrak{b}$ are functions on M.

In addition,

$$\begin{aligned}g(R(\mathfrak{u}_1,\mathfrak{u}_2)\mathfrak{u}_3,\xi)=\eta(R(\mathfrak{u}_1,\mathfrak{u}_2)\mathfrak{u}_3)&=g(\mathfrak{u}_2,\mathfrak{u}_3)\eta(\mathfrak{u}_1)\\&-g(\mathfrak{u}_1,\mathfrak{u}_3)\eta(\mathfrak{u}_2),\end{aligned} \tag{2.9}$$

$$R(\xi,\mathfrak{u}_1)\mathfrak{u}_2=g(\mathfrak{u}_1,\mathfrak{u}_2)\xi-\eta(\mathfrak{u}_2)\mathfrak{u}_1, \tag{2.10}$$

$$R(\xi,\mathfrak{u}_1)\xi=\mathfrak{u}_1+\eta(\mathfrak{u}_2)\xi, \tag{2.11}$$

$$R(\mathfrak{u}_1,\mathfrak{u}_2)\xi=\eta(\mathfrak{u}_2)\mathfrak{u}_1-\eta(\mathfrak{u}_1)\mathfrak{u}_2, \tag{2.12}$$

$$S(\mathfrak{u}_1,\xi)=(n-1)\eta(\mathfrak{u}_1), \tag{2.13}$$

$$\begin{aligned}S(\phi\mathfrak{u}_1,\phi\mathfrak{u}_2)&=S(\mathfrak{u}_1,\mathfrak{u}_2)+(n-1)\eta(\mathfrak{u}_1)\eta(\mathfrak{u}_2),\\&\forall\mathfrak{u}_1,\mathfrak{u}_2\mathfrak{u}_3\in\Im_0^1(M),\end{aligned} \tag{2.14}$$

where R is the Riemannian curvature tensor.

3. Lifts of LP-Sasakian manifolds

Let TM be the tangent bundle of a manifold M and let the function, a 1-form, a vector field and a tensor field type (1,1) be symbolized as $f,\eta,\mathfrak{u}_1$ and ϕ and ∇, respectively. Suppose TM be the tangent bundle and $\mathfrak{u}_1=\mathfrak{u}_1{}^i\frac{\partial}{\partial x^i}$ be a local vector field on M, then its vertical and complete lifts in the term of partial differential equations are

$$\mathfrak{u}_1{}^V=\mathfrak{u}_1{}^i\frac{\partial}{\partial y^i}, \tag{3.1}$$

$$\mathfrak{u}_1{}^C=\mathfrak{u}_1{}^i\frac{\partial}{\partial x^i}+\frac{\partial\mathfrak{u}_1{}^i}{\partial x^j}y^j\frac{\partial}{\partial y^i}. \tag{3.2}$$

The complete and vertical lifts of $f, \eta, \mathfrak{u}_1$ and ϕ are symbolized as $f^C, \eta^C, \mathfrak{u}_1{}^C, \phi^C$ and $f^V, \eta^V, \mathfrak{u}_1{}^V, \phi^V$, respectively. The following functions on $f, \eta, \mathfrak{u}_1$ and ϕ are provided by ([23–25])

$$(f\mathfrak{u}_1)^V = f^V\mathfrak{u}_1{}^V, (f\mathfrak{u}_1)^C = f^C\mathfrak{u}_1{}^V + f^V\mathfrak{u}_1{}^C, \tag{3.3}$$

$$\mathfrak{u}_1{}^V f^V = 0, \mathfrak{u}_1{}^V f^C = \mathfrak{u}_1{}^C f^V = (\mathfrak{u}_1 f)^V, \mathfrak{u}_1{}^C f^C = (\mathfrak{u}_1 f)^C, \tag{3.4}$$

$$\eta^V(f^V) = 0, \eta^V(\mathfrak{u}_1{}^C) = \eta^C(\mathfrak{u}_1{}^V) = \eta(\mathfrak{u}_1)^V, \eta^C(\mathfrak{u}_1{}^C) = \eta(\mathfrak{u}_1)^C, \tag{3.5}$$

$$\phi^V\mathfrak{u}_1{}^C = (\phi\mathfrak{u}_1)^V, \phi^C\mathfrak{u}_1{}^C = (\phi\mathfrak{u}_1)^C, \tag{3.6}$$

$$[\mathfrak{u}_1, \mathfrak{u}_2]^V = [\mathfrak{u}_1{}^C, \mathfrak{u}_2{}^V] = [\mathfrak{u}_1{}^V, \mathfrak{u}_2{}^C], [\mathfrak{u}_1, \mathfrak{u}_2]^C = [\mathfrak{u}_1{}^C, \mathfrak{u}_2{}^C], \tag{3.7}$$

$$\nabla^C_{\mathfrak{u}_1{}^C}\mathfrak{u}_2{}^C = (\nabla_{\mathfrak{u}_1}\mathfrak{u}_2)^C, \quad \nabla^C_{\mathfrak{u}_1{}^C}\mathfrak{u}_2{}^V = (\nabla_{\mathfrak{u}_1}\mathfrak{u}_2)^V, \tag{3.8}$$

where $\mathfrak{u}_1{}^C, \mathfrak{u}_2{}^C \in \Im_0^1(TM)$ and mathematical operators ∇^C and ∇^V are the complete and vertical lifts of ∇ on TM ([26], [27]).

Taking the complete lift by mathematical operators on (2.1)-(2.8), we infer

$$\eta^C(\xi^C) = \eta^V(\xi^V) = 0, \ \eta^C(\xi^V) = \eta^V(\xi^C) = -1, \tag{3.9}$$

$$((\phi^2)^C = I + \eta^C(\mathfrak{u}_1{}^C)\xi^V + \eta^V(\mathfrak{u}_1{}^C)\xi^C, \tag{3.10}$$

$$g^C((\phi\mathfrak{u}_1)^C, (\phi\mathfrak{u}_2)^C) = g^C(\mathfrak{u}_1{}^C, \mathfrak{u}_2{}^C) + \eta^C(\mathfrak{u}_1{}^C)\eta^V(\mathfrak{u}_2{}^C) + \eta^V(\mathfrak{u}_1{}^C)\eta^C(\mathfrak{u}_2{}^C), \tag{3.11}$$

$$g^C(\mathfrak{u}_1{}^C, \xi^C) = \eta^C(\mathfrak{u}_1{}^C), \ \ \nabla^C_{\mathfrak{u}_1{}^C}\xi^C = (\phi\mathfrak{u}_1)^C, \tag{3.12}$$

$$\begin{aligned}(\nabla^C_{\mathfrak{u}_1{}^C}\phi^C)\mathfrak{u}_2{}^C &= g^C(\mathfrak{u}_1{}^C, \mathfrak{u}_2{}^C)\xi^V + g^C(\mathfrak{u}_1{}^V, \mathfrak{u}_2{}^C)\xi^C \\ &+2\{\eta^C(\mathfrak{u}_1{}^C)\eta^C(\mathfrak{u}_2{}^C)\xi^V + \eta^C(\mathfrak{u}_1{}^C)\eta^V(\mathfrak{u}_2{}^C)\xi^C \\ &+\eta^V(\mathfrak{u}_1{}^C)\eta^C(\mathfrak{u}_2{}^C)\xi^C\} + \mathfrak{u}_1{}^C\eta^V(\mathfrak{u}_2{}^C) \\ &+\mathfrak{u}_1{}^V\eta^C(\mathfrak{u}_2{}^C),\end{aligned} \tag{3.13}$$

Also,

$$\begin{aligned}&\phi^C\xi^C = \phi^V\xi^V = \phi^C\xi^V = \phi^V\xi^C = 0 \\ &\eta^C(\phi\mathfrak{u}_1)^C = \eta^V(\phi\mathfrak{u}_1)^V = \eta^C(\phi\mathfrak{u}_1)^V = \eta^V(\phi\mathfrak{u}_1)^C = 0,\end{aligned} \tag{3.14}$$

$$\begin{aligned}S^C(\mathfrak{u}_1{}^C, \mathfrak{u}_2{}^C) &= \mathfrak{a}g^C(\mathfrak{u}_1{}^C, \mathfrak{u}_2{}^C) + \mathfrak{b}\{\eta^C(\mathfrak{u}_1{}^C)\eta^V(\mathfrak{u}_2{}^C) \\ &+\eta^V(\mathfrak{u}_1{}^C)\eta^C(\mathfrak{u}_2{}^C)\},\end{aligned} \tag{3.15}$$

In addition,

$$g^C(R^C(\mathfrak{u}_1{}^C,\mathfrak{u}_2{}^C)\mathfrak{u}_3{}^C,\xi^C)=\eta^C(R^C(\mathfrak{u}_1{}^C,\mathfrak{u}_2{}^C)\mathfrak{u}_3{}^C)$$
$$=g^C(\mathfrak{u}_2{}^C,\mathfrak{u}_3{}^C)\eta^V(\mathfrak{u}_1{}^C)$$
$$+g^C(\mathfrak{u}_2{}^V,\mathfrak{u}_3{}^C)\eta^C(\mathfrak{u}_1{}^C) \tag{3.16}$$
$$-g^C(\mathfrak{u}_1{}^C,\mathfrak{u}_3{}^C)\eta^V(\mathfrak{u}_2{}^C)$$
$$-g^C(\mathfrak{u}_1{}^V,\mathfrak{u}_3{}^C)\eta^C(\mathfrak{u}_2{}^C)$$

$$R^C(\xi^C,\mathfrak{u}_1{}^C)\mathfrak{u}_2{}^C=g^C(\mathfrak{u}_1{}^C,\mathfrak{u}_2{}^C)\xi^V$$
$$+g^C(\mathfrak{u}_1{}^V,\mathfrak{u}_2{}^C)\xi^C-\eta^C(\mathfrak{u}_2{}^C)\mathfrak{u}_1{}^V \tag{3.17}$$
$$-\eta^V(\mathfrak{u}_2{}^C)\mathfrak{u}_1{}^C,$$

$$R^C(\xi^C,\mathfrak{u}_1{}^C)\xi^C=\mathfrak{u}_1{}^C+\eta^C(\mathfrak{u}_2{}^C)\xi^V$$
$$+\eta^V(\mathfrak{u}_2{}^C)\xi^C, \tag{3.18}$$

$$R^C(\mathfrak{u}_1{}^C,\mathfrak{u}_2{}^C)\xi^C=\eta^C(\mathfrak{u}_2{}^C)\mathfrak{u}_1{}^V+\eta^C(\mathfrak{u}_2{}^C)\mathfrak{u}_1{}^V$$
$$-\eta^C(\mathfrak{u}_1{}^C)\mathfrak{u}_2{}^V \tag{3.19}$$
$$-\eta^V(\mathfrak{u}_1{}^C)\mathfrak{u}_2{}^C,$$

$$S^C(\mathfrak{u}_1{}^C,\xi^C)=(n-1)\eta^C(\mathfrak{u}_1{}^C), \tag{3.20}$$

$$S^C((\phi\mathfrak{u}_1)^C,(\phi\mathfrak{u}_2)^C)=S^C(\mathfrak{u}_1{}^C,\mathfrak{u}_2{}^C)$$
$$+(n-1)\{\eta^C(\mathfrak{u}_1{}^C)\eta^V(\mathfrak{u}_2{}^C) \tag{3.21}$$
$$+\eta^V(\mathfrak{u}_1{}^C)\eta^C(\mathfrak{u}_2{}^C)\},$$

$\forall \mathfrak{u}_1{}^C,\mathfrak{u}_2{}^C,\xi^C\in\Im_0^1(TM),\eta^C\in\Im_1^0(TM),R^C,g^C$, S^C and Q^C on TM stand for the complete lifts of R,g,S and Q, respectively.

4. Lifts of LP-Sasakian manifolds with $\mathcal{C}=0$

Let TM be the tangent bundle of an LP-Sasakian manifold M. Taking the complete lift by mathematical operators on (1.1), we infer

$$\mathcal{C}^C(\mathfrak{u}_1{}^C,\mathfrak{u}_2{}^C)\mathfrak{u}_3{}^C=R^C(\mathfrak{u}_1{}^C,\mathfrak{u}_2{}^C)\mathfrak{u}_3{}^C$$
$$-\frac{1}{n-2}\{g^C(\mathfrak{u}_2{}^C,\mathfrak{u}_3{}^C)(Q\mathfrak{u}_1)^V+g^C(\mathfrak{u}_2{}^V,\mathfrak{u}_3{}^C)(Q\mathfrak{u}_1)^C$$
$$-g^C(\mathfrak{u}_1{}^C,\mathfrak{u}_3{}^C)(Q\mathfrak{u}_2)^V-g^C(\mathfrak{u}_1{}^V,\mathfrak{u}_3{}^C)(Q\mathfrak{u}_2)^C$$
$$+S^C(\mathfrak{u}_2{}^C,\mathfrak{u}_3{}^C)\mathfrak{u}_1{}^V+S^C(\mathfrak{u}_2{}^V,\mathfrak{u}_3{}^C)\mathfrak{u}_1{}^C \tag{4.1}$$
$$-S^C(\mathfrak{u}_1{}^C,\mathfrak{u}_3{}^C)\mathfrak{u}_2{}^V-S^C(\mathfrak{u}_1{}^V,\mathfrak{u}_3{}^C)\mathfrak{u}_2{}^C\}$$
$$+\frac{r^C}{(n-1)(n-2)}\{g^C(\mathfrak{u}_2{}^C,\mathfrak{u}_3{}^C)\mathfrak{u}_1{}^V+g^C(\mathfrak{u}_2{}^V,\mathfrak{u}_3{}^C)\mathfrak{u}_1{}^C$$
$$-g^C(\mathfrak{u}_1{}^C,\mathfrak{u}_3{}^C)\mathfrak{u}_2{}^V-g^C(\mathfrak{u}_1{}^V,\mathfrak{u}_3{}^C)\mathfrak{u}_2{}^C\}$$

where

$$S^C(u_1{}^C, u_2{}^C) = g^C((Qu_1)^C, u_2{}^C).$$

Using (1.3) in (4.1), we infer

$$\begin{aligned} R^C(u_1{}^C, u_2{}^C)u_3{}^C = \frac{1}{n-2}\{ & g^C(u_2{}^C, u_3{}^C)(Qu_1)^V + g^C(u_2{}^V, u_3{}^C)(Qu_1)^C \\ & -g^C(u_1{}^C, u_3{}^C)(Qu_2)^V - g^C(u_1{}^V, u_3{}^C)(Qu_2)^C \\ & +S^C(u_2{}^C, u_3{}^C)u_1{}^V + S^C(u_2{}^V, u_3{}^C)u_1{}^C \\ & -S^C(u_1{}^C, u_3{}^C)u_2{}^V - S^C(u_1{}^V, u_3{}^C)u_2{}^C\} \\ & -\frac{r^C}{(n-1)(n-2)}\{g^C(u_2{}^C, u_3{}^C)u_1{}^V + g^C(u_2{}^V, u_3{}^C)u_1{}^C \\ & -g^C(u_1{}^C, u_3{}^C)u_2{}^V - g^C(u_1{}^V, u_3{}^C)u_2{}^C\} \end{aligned} \tag{4.2}$$

Taking $u_3 = \xi$ in (4.1) and using (3.12), (3.19) and (3.20), we find

$$\begin{aligned} &\eta^C(u_2{}^C)u_1{}^V + \eta^V(u_2{}^C)u_1{}^C - \eta^C(u_1{}^C)u_2{}^V - \eta^V(u_1{}^C)u_2{}^C \\ &= \frac{1}{n-2}\{\eta^C(u_2{}^C)(Qu_1)^V + \eta^V(u_2{}^C)(Qu_1)^C \\ &-\eta^C(u_1{}^C)(Qu_2)^V - \eta^V(u_1{}^C)(Qu_2)^C\} \\ &+\frac{n-1}{n-2}\{\eta^C(u_2{}^C)u_1{}^V + \eta^V(u_2{}^C)u_1{}^C \\ &-\eta^C(u_1{}^C)u_2{}^V - \eta^V(u_1{}^C)u_2{}^C\} - \frac{r^C}{(n-1)(n-2)}\{\eta^C(u_2{}^C)u_1{}^V \\ &+\eta^V(u_2{}^C)u_1{}^C - \eta^C(u_1{}^C)u_2{}^V - \eta^V(u_1{}^C)u_2{}^C\}. \end{aligned} \tag{4.3}$$

Taking $u_2 = \xi$ and using (3.9) we wet

$$(Qu_1)^C = \left(\frac{1}{n-1} - 1\right)u_1 + \left(\frac{r^C}{n-1} - 1\right)\{\eta^C(u_1{}^C)\xi^V + \eta^V(u_1{}^C)\xi^C\} \tag{4.4}$$

Hence the manifold is η^C-Einstein on TM. Contracting (4.4) we infer

$$r^C = n(n-1). \tag{4.5}$$

Using (4.5) in (4.4) we find

$$(Qu_1)^C = (n-1)u_1. \tag{4.6}$$

Putting (4.6) in (4.2) we get after a few steps

$$\begin{aligned} R^C(u_1{}^C, u_2{}^C)u_3{}^C = {} & g^C(u_2{}^C, u_3{}^C)u_1{}^V + g^C(u_2{}^V, u_3{}^C)u_1{}^C \\ & -g^C(u_1{}^C, u_3{}^C)u_2{}^V + g^C(u_1{}^V, u_3{}^C)u_2{}^C. \end{aligned} \tag{4.7}$$

This implies that a conformally flat LP-Sasakian manifold on TM is of constant curvature (value=1). Thus we conclude that

Theorem 4.1: *Let TM be the tangent bundle of an LP-Sasakain manifold. Then a conformally flat LP-Sasakian manifold on TM is locally isometric to a unit sphere $S^n(1)$.*

5. Lifts of a Lorentzian para-Sasakian manifolds with $\tilde{\mathcal{C}} = 0$

Let TM be the tangent bundle of an LP-Sasakian manifold M. Applying the complete lift on (1.2), we infer

$$\begin{aligned}\tilde{\mathcal{C}}^C(\mathfrak{u}_1{}^C,\mathfrak{u}_2{}^C)\mathfrak{u}_3{}^C &= \mathfrak{a}R^C(\mathfrak{u}_1{}^C,\mathfrak{u}_2{}^C)\mathfrak{u}_3{}^C\\ &-\mathfrak{b}\{S^C(\mathfrak{u}_2{}^C,\mathfrak{u}_3{}^C)\mathfrak{u}_1{}^V + S^C(\mathfrak{u}_2{}^V,\mathfrak{u}_3{}^C)\mathfrak{u}_1{}^C\\ &-S^C(\mathfrak{u}_1{}^C,\mathfrak{u}_3{}^C)\mathfrak{u}_2{}^V - S^C(\mathfrak{u}_1{}^V,\mathfrak{u}_3{}^C)\mathfrak{u}_2{}^C\\ &+g^C(\mathfrak{u}_2{}^C,\mathfrak{u}_3{}^C)(Q\mathfrak{u}_1)^V + g^C(\mathfrak{u}_2{}^V,\mathfrak{u}_3{}^C)(QX)^C\\ &-g^C(\mathfrak{u}_1{}^C,\mathfrak{u}_3{}^C)(Q\mathfrak{u}_2)^V - g^C(\mathfrak{u}_1{}^V,\mathfrak{u}_3{}^C)(Q\mathfrak{u}_2)^C\}\\ &-\frac{r^C}{n}\left(\frac{\mathfrak{a}}{n-1}+2\mathfrak{b}\right)\{g^C(\mathfrak{u}_2{}^C,\mathfrak{u}_3{}^C)\mathfrak{u}_1{}^V + g^C(\mathfrak{u}_2{}^V,\mathfrak{u}_3{}^C)\mathfrak{u}_1{}^C\\ &-g^C(\mathfrak{u}_1{}^C,\mathfrak{u}_3{}^C)\mathfrak{u}_2{}^V - g^C(\mathfrak{u}_1{}^V,\mathfrak{u}_3{}^C)\mathfrak{u}_2)^C\}\end{aligned} \tag{5.1}$$

Applying (1.4) in above equation, we infer

$$\begin{aligned}R^C(\mathfrak{u}_1{}^C,\mathfrak{u}_2{}^C)\mathfrak{u}_3{}^C &= -\frac{\mathfrak{a}}{\mathfrak{b}}\{S^C(\mathfrak{u}_2{}^C,\mathfrak{u}_3{}^C)\mathfrak{u}_1{}^V + S^C(\mathfrak{u}_2{}^V,\mathfrak{u}_3{}^C)\mathfrak{u}_1{}^C\\ &-S^C(\mathfrak{u}_1{}^C,\mathfrak{u}_3{}^C)\mathfrak{u}_2{}^V - S^C(\mathfrak{u}_1{}^V,\mathfrak{u}_3{}^C)\mathfrak{u}_2{}^C\\ &+g^C(\mathfrak{u}_2{}^C,\mathfrak{u}_3{}^C)(Q\mathfrak{u}_1)^V + g^C(\mathfrak{u}_2{}^V,\mathfrak{u}_3{}^C)(Q\mathfrak{u}_1)^C\\ &-g^C(\mathfrak{u}_1{}^C,\mathfrak{u}_3{}^C)(Q\mathfrak{u}_2)^V - g^C(\mathfrak{u}_1{}^V,\mathfrak{u}_3{}^C)(Q\mathfrak{u}_2)^C\}\\ &+\frac{r^C}{n}\left(\frac{\mathfrak{a}}{n-1}+2\mathfrak{b}\right)\{g^C(\mathfrak{u}_2{}^C,\mathfrak{u}_3{}^C)\mathfrak{u}_1{}^V + g^C(\mathfrak{u}_2{}^V,\mathfrak{u}_3{}^C)\mathfrak{u}_1{}^C\\ &-g^C(\mathfrak{u}_1{}^C,\mathfrak{u}_3{}^C)\mathfrak{u}_2{}^V - g^C(\mathfrak{u}_1{}^V,\mathfrak{u}_3{}^C)\mathfrak{u}_2)^C\}.\end{aligned} \tag{5.2}$$

Taking $\mathfrak{u}_3 = \xi$ in (5.2) and using (3.12), (3.19) and (3.20), we find

$$\begin{aligned}&\eta^C(\mathfrak{u}_2{}^C)\mathfrak{u}_1{}^V + \eta^V(\mathfrak{u}_2{}^C)\mathfrak{u}_1{}^C - \eta^C(\mathfrak{u}_1{}^C)\mathfrak{u}_2{}^V - \eta^V(\mathfrak{u}_1{}^C)\mathfrak{u}_2{}^C\\ &= \frac{\mathfrak{b}}{\mathfrak{a}}\{\eta^C(\mathfrak{u}_2{}^C)(Q\mathfrak{u}_1)^V + \eta^V(\mathfrak{u}_2{}^C)(Q\mathfrak{u}_1)^C\\ &-\eta^C(\mathfrak{u}_1{}^C)(Q\mathfrak{u}_2)^V - \eta^V(\mathfrak{u}_1{}^C)(Q\mathfrak{u}_2)^C\}\\ &+\left(\frac{r^C}{\mathfrak{a}n}\left(\frac{\mathfrak{a}}{n-1}+2\mathfrak{b}\right) - \frac{\mathfrak{b}}{\mathfrak{a}}(n-1)\right)\\ &\{\eta^C(\mathfrak{u}_2{}^C)\mathfrak{u}_1{}^V + \eta^V(\mathfrak{u}_2{}^C)\mathfrak{u}_1{}^C - \eta^C(\mathfrak{u}_1{}^C)\mathfrak{u}_2{}^V - \eta^V(\mathfrak{u}_1{}^C)\mathfrak{u}_2{}^C\}.\end{aligned} \tag{5.3}$$

Setting $\mathfrak{u}_2 = \xi$ and using (3.9) we infer

$$(Q\mathfrak{u}_1)^C = \left(\frac{r^C}{\mathfrak{b}n}\left(\frac{\mathfrak{a}}{n-1}+2\mathfrak{b}\right)-(n-1)-\frac{\mathfrak{b}}{\mathfrak{a}}\right)\mathfrak{u}_1{}^C$$
$$+\left(\frac{r^C}{\mathfrak{b}n}\left(\frac{\mathfrak{a}}{n-1}+2\mathfrak{b}\right)-\frac{\mathfrak{a}}{\mathfrak{b}}-2(n-1)\right)\{\eta^C(\mathfrak{u}_1{}^C)\xi^V+\eta^V(\mathfrak{u}_1{}^C)\xi^C\}. \quad (5.4)$$

Contracting (5.4), we infer

$$r^C = n(n-1). \quad (5.5)$$

Making use of (5.5) in (5.4), we infer

$$(Q\mathfrak{u}_1)^C = (n-1)\mathfrak{u}_1{}^C. \quad (5.6)$$

On applying (5.6) in (5.2), equation (5.2) becomes

$$R^C(\mathfrak{u}_1{}^C,\mathfrak{u}_2{}^C)\mathfrak{u}_3{}^C = g^C(\mathfrak{u}_2{}^C,\mathfrak{u}_3{}^C)\mathfrak{u}_1{}^V + g^C(\mathfrak{u}_2{}^V,\mathfrak{u}_3{}^C)\mathfrak{u}_1{}^C$$
$$-g^C(\mathfrak{u}_1{}^C,\mathfrak{u}_3{}^C)\mathfrak{u}_2{}^V - g^C(\mathfrak{u}_1{}^V,\mathfrak{u}_3{}^C)\mathfrak{u}_2{}^C.$$

Thus we conclude that

Theorem 5.1: *Let TM be the tangent bundle of an LP-Sasakain manifold. Then a quasi conformally flat LP-Sasakian manifold on TM is locally isometric with a unit sphere $S^n(1)$.*

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